Absolute continuity of harmonic measure on rough domains

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May 9

Rainwater Seminar - University of Washington
Part I

Detecting the exit point of Brownian motion
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**Dirichlet problem:**

(D) \[ \begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = f \text{ on } \partial \Omega \\ u \in C^2(\Omega) \cap C(\partial \Omega) \\ f \in C_c(\partial \Omega). \end{cases} \]

$\Delta := \partial_{x_1x_1} + \ldots + \partial_{x_{n+1}x_{n+1}}$
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**Dirichlet problem:**

**Potential Theory:** \( \exists! \) a family of probability measures \( \{\omega_X^\Omega\}_{X \in \Omega} \) on \( \partial \Omega \) called harmonic measure of \( \Omega \) with a pole at \( X \in \Omega \) such that

\[ u(X) = \int_{\partial \Omega} f(Q) \, d\omega_X^\Omega(Q) \text{ solves } (D). \]
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**Potential Theory:** There exists a family of probability measures $\{\omega^X_\Omega\}_{X \in \Omega}$ on $\partial \Omega$ called harmonic measure of $\Omega$ with a pole at $X \in \Omega$ such that

\[u(X) = \int_{\partial \Omega} f(Q) \, d\omega^X_\Omega(Q) \quad \text{solves (D)}.
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**Probability:** Harmonic measure $\omega^X_\Omega(E)$ of $E$ with a given pole $X$ is the probability that a Brownian motion starting at $X$ will first hit $\partial \Omega$ in the set $E$. 
Examples of Harmonic Measure

- If $\Omega \subset \mathbb{R}^2$ is a simply connected domain, $\partial \Omega$ is Jordan curve

![Diagram showing conformal transformation and harmonic measure](image)
If $\Omega \subset \mathbb{R}^2$ is a simply connected domain, $\partial \Omega$ is Jordan curve, then by Carathéodory’s theorem

$$\omega^x_\Omega(E) = \frac{\text{arclength}(\psi^{-1}(E))}{2\pi}$$
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- If $\Omega = A(0, r, R) \subset \mathbb{R}^{n+1}$ is an annular region then the harmonic measure of the inner shell $S(0, r)$ is

$$\omega^X_\Omega(S(0, r)) = \begin{cases} \rac{\log R - \log |X|}{\log R - \log r} & \text{if } n = 1, \\ \frac{|X|^{2-(n+1)} - R^{2-(n+1)}}{r^{2-(n+1)} - R^{2-(n+1)}} & \text{if } n \geq 2. \end{cases}$$
Assume $\Omega$ is at least $C^1$ and bounded. Let $K_\Omega(X, \xi)$ be the Poisson kernel for $\Omega$ and $E \subset \partial \Omega$;

\[
\omega^X_\Omega(E) = \int \chi_E(\xi) K_\Omega(X, \xi) d\mathcal{H}^n(\xi).
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More examples of harmonic Measure

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**Example**

Let $\Omega = \mathbb{R}^2_+$ and $E = [-T, T] \times \{0\}$, $z = x + iy$. Find $\omega^z_{\Omega}(E)$. 
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$$\omega_\Omega^z(E) = \int_{\partial \Omega} \chi_{[-T, T]}(t) \frac{1}{\pi} \frac{y}{(x - t)^2 + y^2} dt = \int_{-T}^{T} \frac{1}{\pi} \frac{y}{(x - t)^2 + y^2} dt$$

$$= \frac{1}{\pi} \arctan \left( \frac{x + T}{y} \right) - \frac{1}{\pi} \arctan \left( \frac{x - T}{y} \right)$$
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Notice that $\omega^z_\Omega(E)$ is a harmonic function and

$$\begin{cases} 
\omega^z_\Omega(E) \to 1 & \text{as } z \to E \subset \partial \Omega, \\
\omega^z_\Omega(E) \to 0 & \text{as } z \to \partial \Omega \setminus E.
\end{cases}$$
Even more examples of Harmonic Measures

- If $\Omega = \mathbb{B}^{n+1}$, $(n + 1)$-dimensional unit ball, and $X \in \Omega$. Then

$$\omega^X(E) = \frac{1}{\mathcal{H}^n(S^n)} \int_E \frac{1 - |X|^2}{|X - Y|^{n+1}} \, d\mathcal{H}^n(Y) \quad \text{for every Borel set } E \subset S^n.$$
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  \]

- When the pole $X = 0$ and $\Omega = \mathbb{B}^{n+1}$ then
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  \omega^0(E) = \frac{\mathcal{H}^n(E)}{\mathcal{H}^n(\mathbb{S}^n)} \quad \text{for every Borel set } E \subset \mathbb{S}^n.
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$\mathcal{H}^n$ is the $n -$dimensional Hausdorff measure which will be defined soon.
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- If \( \Omega \subset \mathbb{R}^{n+1} \) is bounded domain of class \( C^1 \), then there is \( K(X, Y) : \Omega \times \partial \Omega \to \mathbb{R} \) such that

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\omega^X(E) = \int_E K(X, Y) d\mathcal{H}^n(Y) \quad \text{for every Borel set } E \subset \partial \Omega.
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\( \mathcal{H}^n \) is the \( n \)–dimensional Hausdorff measure which will be defined soon.
For every Borel set $E \subset \partial \Omega$, $X \to \omega^X(E)$ is a non-negative harmonic function in $\Omega$. 

Drop the pole $X$ to get the harmonic measure $\omega$. Therefore, the sets of harmonic measure zero do not depend on the pole.
For every Borel set $E \subset \partial \Omega$, $X \rightarrow \omega^X(E)$ is a non-negative harmonic function in $\Omega$.

Harmonic measure $\omega^{X_1}$ and $\omega^{X_2}$ at different poles are mutually absolutely continuous; $\omega^{X_1}(E) = 0 \Leftrightarrow \omega^{X_2}(E) = 0$.

$$c^{-1} \omega^{X_1}(E) \leq \omega^{X_2}(E) \leq c \omega^{X_1}(E).$$
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$C^{-1}\omega^X_1(E) \leq \omega^X_2(E) \leq C\omega^X_1(E)$.

Therefore, the sets of harmonic measure zero do not depend on the pole.
Detecting the exit point of a Brownian motion

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- Aim is to find the point where it first hits the boundary $\partial \Omega$. 
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If a detector of radius $r$ costs us $\phi(r)$ (for some increasing $\phi$ on $(0, \infty)$), can we detect the exit point almost surely on a finite budget?
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- Note that to detect an exit at $x$, the point must be contained in infinitely many detectors whose radii tend to zero.
Detecting the exit point of Brownian motion

► When $\Omega$ is the unit disk $\mathbb{D}$, and the Brownian particle starts at 0 then the hitting distribution on $\partial \Omega$ is normalized Lebesgue measure.

► Thus to detect the exit point almost surely, we must cover almost every point of $\partial \Omega$ by arbitrarily small balls.
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- If $\phi(r) \geq r$ then we can not detect the exit point on a finite budget.

- However, if $\phi(r) = o(r)$ then we can cover $\partial \Omega$ by about $n_k$ balls of size $1/n_k$ and let $n_k \uparrow \infty$ so fast that $\sum n_k \phi(1/n_k) < \infty$. 
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- If \( \partial \Omega \) is the von Koch Snowflake then it takes roughly \( 4^n \) balls of size \( 3^n \) to cover the whole boundary, which we can do on a finite budget iff \( \phi(t) = o(t^\alpha) \), where \( \alpha = \log 4 / \log 3 > 1 \).
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- However, not all parts of the snowflake are equally likely to be hit by Brownian motion, and there is a small subset of $\partial\Omega$ which still gets hit with probability 1.
Hausdorff measure and Hausdorff dimension

We estimate harmonic measure by comparing it to the more geometrically defined Hausdorff measures:

\[(H^\alpha)(E) = \lim_{\delta \to 0} \inf \left( \sum_{i=1}^{\infty} r_i \right) \]

When \( \alpha = 1 \) we then denote this by \( H_1(E) = \lim_{\delta \to 0} \inf \left( \sum_{i=1}^{\infty} r_i \right) \).

\( H_2 \) is multiple of Lebesgue area measure; \( H_1 \) is length...
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Let $\phi$ be increasing function on $[0, \infty)$,

$$\mathcal{H}_\phi(E) := \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} \phi(r_i); \; E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), \; r_i \leq \delta \right\}.$$
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- When $\phi(t) = t^\alpha$ we then denote this by $\mathcal{H}^\alpha$;

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\( \mathcal{H}^n_\infty(B) \) is called the Hausdorff content of \( E \) and is defined as

\[
\mathcal{H}^n_\infty(E) = \inf \left\{ \sum_{i=1}^{\infty} (r_i)^n; \ E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i) \right\}.
\]

- \( \mathcal{H}^\alpha_\infty(E) \leq \mathcal{H}^\alpha_\delta(E) \leq \mathcal{H}^\alpha(E) \). But still \( \mathcal{H}^\alpha_\infty(E) = 0 \iff \mathcal{H}^\alpha(E) = 0 \).
Being singular $\perp$ – absolutely continuous $\Leftrightarrow$

The Hausdorff dimension of a set $E$ is defined by

$$\dim_{\mathcal{H}}(E) = \inf\{\alpha; \mathcal{H}^\alpha(E) = 0\}.$$  

The smaller $\alpha$ is, the more expensive it is to cover $E$; the dimension marks the transition from positive to zero cost coverings.
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The dimension of a measure $\mu$ is the smallest dimension of a full $\mu$-measure set, i.e.,

$$\dim_H(\mu) = \inf\{\dim_H(E) : \mu(E^c) = 0\} = \inf\{\alpha : \mu \perp \mathcal{H}^\alpha\}$$

- $\mu \perp \nu$ if there is a set $E$ such that $\mu(E) = \nu(E^c) = 0$
- $\mu \ll \nu$ if $\nu(E) = 0 \Rightarrow \mu(E) = 0$.
- $\mu \sim \nu$ if $\nu \ll \mu \ll \nu$. 

It is always true that $\dim_H(\mu) \leq \dim_H(\text{supp}(\mu))$. 

Thus the detection question is really: For which we have $\not\perp_H \perp$ and when is $\not\perp_H \perp \ll_H \ll$. 


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Main question and the first result

$\Rightarrow n - 1 \leq \dim_H(\omega) < n + 1$ (in fact $\mathcal{H}^{n-1}(E) = 0 \Rightarrow \omega(E) = 0$).
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- $n - 1 \leq \dim_{\mathcal{H}}(\omega) < n + 1$ (in fact $\mathcal{H}^{n-1}(E) = 0 \Rightarrow \omega(E) = 0$).

**Question**

*Find sufficient conditions (geometric and/or analytic) on $\Omega$ for which we have $\omega \ll \mathcal{H}^n$ on $\partial\Omega$?*
Main question and the first result

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Question

Find sufficient conditions (geometric and/or analytic) on \( \Omega \) for which we have

\[ \omega \ll \mathcal{H}^n \text{ on } \partial \Omega? \]

Theorem (F. and M. Riesz(1916))

Let \( \Omega \) be a simply connected domain in the plane with \( \mathcal{H}^1(\partial \Omega) < \infty \). Let \( \psi : \mathbb{D} \to \Omega \) be conformal.

Then \( \psi' \in L^1(\partial \mathbb{D}) \). Moreover, for any Borel set \( E \subset \partial \mathbb{D} \),

\[ \mathcal{H}^1(\psi(E)) = \int_E |\psi'(e^{i\theta})| \, d\theta. \]

Hence, using \( \omega^2_{\Omega}(K) = 1/2\pi \text{ arclength}(\psi^{-1}(K)) \), \( K \subset \partial \Omega \), one has

\[ \omega(A) = 0 \iff \mathcal{H}^1(A) = 0 \text{ whenever } A \subset \partial \Omega \text{ Borel.} \]

i.e. \( \omega \ll \mathcal{H}^1 \ll \omega \text{ on } \partial \Omega. \)
Thanks!
Detecting the exit point of Brownian motion

How does our problem about the cost of detecting exit points fit into this notation?
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A collection $\mathcal{C}$ of balls is called a Vitali covering of a set $E$ if for each $\varepsilon > 0$,

$$\mathcal{C}_\varepsilon = \{ D \in \mathcal{C} : \text{diam}(D) < \varepsilon \}$$

is also a cover.

We can detect a.e. exit point of Brownian motion on a finite $\phi$-budget iff there is a Vitali covering of a full $\omega$-measure set $E$ by balls of radius $\{r_j\}$ such that $\sum \phi(r_j) < \infty$.
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This happens iff $\mathcal{H}_\phi(E) = 0$ which happens iff $\omega \perp \mathcal{H}_\phi$. 
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This happens iff $\mathcal{H}_\phi(E) = 0$ which happens iff $\omega \perp \mathcal{H}_\phi$.

Conversely $\omega \ll \mathcal{H}_\phi$ holds iff any set $E$ which we can afford to test has zero $\omega$-measure.
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is also a cover.

We can detect a.e. exit point of Brownian motion on a finite $\phi$-budget iff there is a Vitali covering of a full $\omega$-measure set $E$ by balls of radius $\{r_j\}$ such that $\sum \phi(r_j) < \infty$.

This happens iff $\mathcal{H}_\phi(E) = 0$ which happens iff $\omega \perp \mathcal{H}_\phi$.

- Conversely $\omega \ll \mathcal{H}_\phi$ holds iff any set $E$ which we can afford to test has zero $\omega$-measure.

Thus the detection question is really:

- For which $\phi$ we have $\omega \perp \mathcal{H}_\phi$ and when is $\omega \ll \mathcal{H}_\phi$?
Part II

Absolute continuity of harmonic measure on rough domains
Question

Find sufficient conditions for $\omega \ll \mathcal{H}^n$ on $\partial \Omega$?
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- **F. and M. Riesz** (’16): If $\Omega \subset \mathbb{C}$ is domain bounded by Jordan curve of finite length then $\omega \ll \mathcal{H}^1 \ll \omega$ on $\partial \Omega$ (i.e., $\omega \sim \mathcal{H}^1$).
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- **McMillan** (‘69): Let $\Omega \subset \mathbb{C}$ be simply connected and $E \subset \partial \Omega$ be cone points of $\Omega$ then $\omega \sim \mathcal{H}^1$ on $E$. 
Necessary and sufficient conditions for absolute continuity

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- **Makarov** ('85): If $\Omega$ is simply connected then $\dim_H(\omega) = 1$, i.e., $\omega \ll H_\phi$ where $\phi(r) = r e^{A \sqrt{\log \frac{1}{r} \log \log \log \frac{1}{r}}}$ for some $A >> 1$.

- **Bishop and Jones** ('90): Let $\Omega \subset \mathbb{C}$ be simply connected. Then $\omega \ll H^1$ on $\partial \Omega$ whenever $\gamma$ is curve of finite length.

- **Wu** ('86): $\omega \not\ll H^2$ for some topological sphere in $\mathbb{R}^3$.

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Necessary and sufficient conditions for absolute continuity

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Non-tangentially Accessible Domains (NTA)

- Openness

\[ \partial \Omega \]

\[ \Omega \]

\[ \Omega \text{ is NTA} \]
Non-tangentially Accessible Domains (NTA)

- Openness $\leadsto$ Corkscrew condition.

$\partial \Omega$ is NTA if $\partial \Omega$ is Ahlfors regular (AR) and $\Omega$ is interior corkscrew and Harnack chain.

Exterior Corkscrew.

$\partial \Omega$ is $\mathcal{X}$ if $\partial \Omega \cap B(z, r)$ for all $z \in \partial \Omega$ and $r \in (0, \text{diam}(\partial \Omega))$. 
Non-tangentially Accessible Domains (NTA)

- Path-connectedness

\[ \partial \Omega \]

\[ \Omega \]
Non-tangentially Accessible Domains (NTA)

- Path-connectedness $\rightsquigarrow$ Harnack chain condition.
Non-tangentially Accessible Domains (NTA)

- Openness $\sim \rightarrow$ Corkscrew condition.
- Path-connectedness $\sim \rightarrow$ Harnack chain condition.

$\Omega$ is NTA $\equiv$ \begin{cases}  
  \textbf{Interior} & \text{Corkscrew and Harnack Chain.} \\
  \textbf{Exterior} & \text{Corkscrew.} 
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$\Omega$ is NTA $\equiv \begin{cases} 
\textbf{Interior} & \text{Corkscrew and Harnack Chain.} \\
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\end{cases}$

- $\partial \Omega$ is $n-$Ahlfors regular (AR) if

$$cr^n \leq \mathcal{H}^n(\partial \Omega \cap B(z, r)) \leq cr^n \text{ whenever } z \in \partial \Omega \text{ and } r \in (0, \text{diam}(\partial \Omega)).$$
Examples of such domains

- Smooth Domains
- Lipschitz Domains
- NTA Domains

- NTA domains need not be graph domains or of finite perimeter.
Rectifiability of a set

Let $\Sigma = f(\mathbb{R}^n)$ be a Lipschitz image of $\mathbb{R}^n$. 
Rectifiability of a set

Let $\Sigma = f(\mathbb{R}^n)$ be a Lipschitz image of $\mathbb{R}^n$.

$\implies E \subset \mathbb{R}^{n+1}$ is $n$–rectifiable if there exists a family $\{\Sigma_i\}_i$ of Lipschitz images of $\mathbb{R}^n$ such that

\[
E \subset \left( \bigcup_{i=1}^{\infty} \Sigma_i \right) \cup \Sigma_0 \quad \text{with } \mathcal{H}^n(\Sigma_0) = 0.
\]

That is,

\[
\mathcal{H}^n \left( E \setminus \bigcup_{i=1}^{\infty} \Sigma_i \right) = 0,
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$E \subset \mathbb{R}^{n+1}$ is $n$–purely unrectifiable if $E$ contains NO $n$–rectifiable set $F$ with $\mathcal{H}^n(F) > 0$. 

[$\text{Besicovitch-Federer}$]

$E$ is purely unrectifiable if $0 < \mathcal{H}^n(E) < 1$ and $\mathcal{H}^n(\Sigma) = 0$ for almost every $n$–dimensional plane $\Sigma \subset \mathbb{R}^{n+1}$.
Rectifiability of a set

Let $\Sigma = f(\mathbb{R}^n)$ be a Lipschitz image of $\mathbb{R}^n$.

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  \[
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- $E \subset \mathbb{R}^{n+1}$ is $n$–purely unrectifiable if $E$ contains NO $n$–rectifiable set $F$ with $H^n(F) > 0$.

- [Besicovitch-Federer] $E$ is $n$–purely unrectifiable if $0 < H^n(E) < \infty$ and $H^n(\pi_L(E)) = 0$ for almost every $n$–dimensional plane $L \subset \mathbb{R}^{n+1}$.

Here $\pi_L$ denotes the orthogonal projection of $\mathbb{R}^{n+1}$ onto $L$. 
An example of a purely unrectifiable set

The usual example is 4-corner Cantor set.
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\[ C_0 \quad C_1 \quad C_2 \quad \cdots \quad C_{\infty} = \bigcap_k C_k \]

There exists \( c > 1 \) such that for each \( z \in C_1 \) and \( r \in (0, \frac{1}{c}) \),

\[ c r \leq H^1 \left( C_1 \setminus B(z, r) \right) \leq c r \]

For almost every line \( L \) in \( \mathbb{R}^2 \),

\[ H^1 \left( \pi_L \left( C_1 \right) \right) = 0 \]

Hence \( C_1 \) is a purely 1-unrectifiable set.

Every rectifiable curve intersects \( C_1 \) in a set of zero \( H^1 \)-measure.
An example of a purely unrectifiable set

The usual example is 4-corner Cantor set.

\[ C_0 \cap C_1 \cap C_2 = \bigcap_k C_k \]

There exists \( c > 1 \) such that for each \( z \in C_\infty \) and \( r \in (0, \sqrt{2}) \)

\[ c^{-1} r \leq \mathcal{H}^1(C_\infty \cap B(z, r)) \leq cr \]
An example of a purely unrectifiable set

The usual example is 4-corner Cantor set.

$C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots \supseteq C_{\infty} = \bigcap_k C_k$

- There exists $c > 1$ such that for each $z \in C_{\infty}$ and $r \in (0, \sqrt{2})$,

  $$c^{-1} r \leq \mathcal{H}^1(C_{\infty} \cap B(z, r)) \leq cr$$

- For almost every line $L$ in $\mathbb{R}^2$, $\mathcal{H}^1(\pi_L(C_{\infty})) = 0$.

- Hence $C_{\infty}$ is a purely 1-unrectifiable.

- Every rectifiable curve intersects $C_{\infty}$ in a set of zero $\mathcal{H}^1$-measure.
Global results in higher dimension

- **Dahlberg** ('77): If $\partial \Omega$ is a union of Lipschitz graphs then $\omega \sim H^n$ on $\partial \Omega$. 

- **Wu** ('86): Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain satisfying exterior corkscrew condition and suppose that $\omega$ is an $A_{\infty}$ set and divides $\mathbb{R}^{n+1}$ into two NTA domains. Then $\omega \sim H^n$ on $\Omega \setminus \partial \Omega$.

- **David and Jerison** ('90); **Semmes** ('89): If $\Omega$ is NTA and $\partial \Omega$ is AR then $\omega \sim H^n$ on $\partial \Omega$.

- **Badger** ('12): If $\Omega \subset \mathbb{R}^{n+1}$ is NTA with $H^n(\partial \Omega) < 1$ then $\omega \sim H^n$ on $\Omega \setminus \partial \Omega$; \[ \liminf_{r \to 0} \frac{H^n(\partial \Omega \setminus B(x, r))}{r^n} < 1 \].

- **Azzam, Mourgoglou, and Tolsa** ('15): There exists a NTA domain $\Omega$ with $H^n(\partial \Omega) < 1$ such that $\omega \not\sim H^n$ on $\partial \Omega$.

(Using the deep result of Wolff which was further developed by Lewis, Nyström, Vogel).
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- **Azzam, Mourgoglou, and Tolsa ('15):** $\exists$ NTA domain $\Omega$ with $\mathcal{H}^n(\partial \Omega) < \infty$ such that $\omega \ll \mathcal{H}^n|_{\partial \Omega}$ (Using the deep result of Wolff which was further developed by Lewis, Nyström, Vogel).
Necessary conditions for Absolute Continuity

- Pommerenke ('86): If $\Omega \subset \mathbb{C}$ is simply connected and $\omega \ll \mathcal{H}^1$ on a set $E \subset \partial \Omega$ then $\omega$ a.e. point $E$ is a cone point for $\Omega$ and $\omega$—almost all of $F$ can be covered by a countable union of 1—dimensional (possibly rotated) Lipschitz graphs (i.e., $\omega|_E$ is rectifiable).
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- Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, open and connected.
- Let $F \subset \partial \Omega$ with $\mathcal{H}^n(F) < \infty$.

1. If $\omega \ll \mathcal{H}^n$ on $F$ $\implies$ $\omega|_F$ is $n$–rectifiable.
2. If $\mathcal{H}^n \ll \omega$ on $F$ $\implies$ $F$ is $n$–rectifiable.

* Portion of the boundary should be contained in a nice rectifiable set (like a graph or curve)!
Notion of Capacity - First Definition

Let $E$ be a closed subset of $\mathbb{R}^{n+1}, n \geq 2$. Then

$$\text{Cap}(E) = \inf \left\{ \int |\nabla v|^2 dx, \ v \in C_0^\infty(\mathbb{R}^{n+1}), \ v \geq 1 \text{ on } E \right\}.$$
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$$\text{Cap}(E) = \inf \left\{ \int |\nabla v|^2 dx, \ v \in C_0^\infty(\mathbb{R}^{n+1}), \ v \geq 1 \text{ on } E \right\}.$$ 

If $u$ is the minimizer of energy then $u$ weakly satisfies

$$\begin{cases} 
\Delta u = 0 & \text{in } \mathbb{R}^{n+1} \setminus E, \\
u = 1 & \text{on } E, \\
u \to 0 & \text{as } |x| \to \infty.
\end{cases}$$
Let $E$ be a closed subset of $\mathbb{R}^{n+1}$, $n \geq 2$. Then

$$\text{Cap}(E) = \inf \left\{ \int |\nabla v|^2 \, dx, \ v \in C^\infty_0(\mathbb{R}^{n+1}), \ v \geq 1 \text{ on } E \right\}.$$

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Indeed, if $u$ solves abode Dirichlet problem then

$$u(x) = \gamma c_n |x|^{2-(n+1)} + o(|x|^{1-(n+1)}) \text{ for } |x| \to \infty.$$
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Indeed, if $u$ solves abode Dirichlet problem then

$$u(x) = \gamma c_n |x|^{2-(n+1)} + o(|x|^{1-(n+1)}) \text{ for } |x| \to \infty.$$ 

Then,

$$\text{Cap}(E) = \gamma = \lim_{|x| \to \infty} \frac{u(x)}{|x|^{(n+1)-2}}.$$ 

This definition is called the electrostatic capacity of $E$. 
Potential of a given measure $\mu$ is defined as

$$U_\mu(x) = \int_E \frac{1}{|x-y|^{(n+1)-2}} d\mu(y)$$
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$$\mathrm{Cap}(E) = \sup\{\mu(E); U_\mu(x) \leq 1, \ x \in \text{supp}(\mu)\}.$$ 

Then $\mathrm{Cap}(E) = \nu(E)$ for a measure $\nu$ which is called equilibrium measure and satisfying $U_\nu(x) \leq 1$ for $x \in \text{supp}(\nu)$ and $U_\nu(x) \geq 1$ up to a set of measure zero capacity on $E$. Note that $U_\nu$ is a positive super harmonic function in $\mathbb{R}^{n+1}$ and harmonic outside of $E$. 
Notion of Capacity - Second Definition

Potential of a given measure $\mu$ is defined as

$$U_\mu(x) = \int_E \frac{1}{|x - y|^{(n+1)-2}} d\mu(y)$$

Then

$$\text{Cap}(E) = \sup\{\mu(E); \ U_\mu(x) \leq 1, \ x \in \text{supp}(\mu)\}.$$ 

Then $\text{Cap}(E) = \nu(E)$ for a measure $\nu$ which is called equilibrium measure and satisfying $U_\nu(x) \leq 1$ for $x \in \text{supp}(\nu)$ and $U_\nu(x) \geq 1$ up to a set of measure zero capacity on $E$. Note that $U_\nu$ is a positive super harmonic function in $\mathbb{R}^{n+1}$ and harmonic outside of $E$. Also

$$\text{Cap}(E) = \left[ \inf\{\mathcal{K}_\mu: \ \mu(E) = 1, \ \mu(E^c) = 0\} \right]^{-1}$$

where

$$\mathcal{K}_\mu = \iint_{E \times E} \frac{1}{|x - y|^{(n+1)-2}} d\mu(x)d\mu(y).$$

which denotes the energy of $\mu$ with respect to the kernel $1/|x|^{(n+1)-2}$. 
Counter example of Wu revisited.

In $\mathbb{R}^2$, there exists simply connected Jordan domain $K$ satisfying

1. $K \cap \{x : x_1 > 0\} \subset \{x : |x| < 2\}$, $K \cap \{x : x_1 < 0\} = \{x : x_1 < 0, |x| < 3\}$
2. $\partial_2 K$ has Hausdorff dimension 1,
3. $\text{Cap}_3(\partial_2 K) > 0$,
4. $\text{Cap}_3(K_\epsilon) \to 0$ as $\epsilon \to 0$ where $K_\epsilon = \{x \in K : \text{dist}(x, \partial_2 K) < \epsilon\}$. 

Let $\bar{\Omega} = B(0, 20) \cap \bar{K}$ in $\mathbb{R}^3$. Then $\!\bar{\Omega} (\partial_2 K) > 0 = H_2(\partial_2 K)$.

Key point here is that for $0 < \beta < \epsilon$, $\text{Cap}_3(K_\epsilon \cap \bar{K} \cap \partial_2 K) < 10 \cdot \text{Cap}_3(\partial_2 K)$. 

Counter example of Wu revisited.

In $\mathbb{R}^2$, there exists simply connected Jordan domain $K$ satisfying

1. $K \cap \{x : x_1 > 0\} \subset \{x : |x| < 2\}$, $K \cap \{x : x_1 < 0\} = \{x : x_1 < 0, |x| < 3\}$
2. $\partial_2 K$ has Hausdorff dimension 1,
3. $\text{Cap}_3(\partial_2 K) > 0$,
4. $\text{Cap}_3(K_\epsilon) \to 0$ as $\epsilon \to 0$ where $K_\epsilon = \{x \in K : \text{dist}(x, \partial_2 K) < \epsilon\}$. 

Let $\Omega$ any set satisfying (1)-(4). Identify the set $\{x, 0) \times K\}$ in $\mathbb{R}^3$. Let $\Omega' = B(0, 20) \cap \overline{K}$ in $\mathbb{R}^3$. Then $\Omega' \cap (\partial_2 K)$ has Hausdorff dimension 1.
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![Diagram of a simply connected Jordan domain K in R^2 with a shaded area and two vertical bars on the boundary]
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![Diagram of a simply connected Jordan domain K](image-url)
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Key point here is that for $0 < \eta < \epsilon$, $\text{Cap}_3(K_\epsilon \setminus \bar{K}_\eta) < \frac{1}{100} \text{Cap}_3(\partial_2 K)$.
Necessary condition for $\omega \ll H^n$: Portion of the boundary should be contained in a nice rectifiable set (like a graph or curve).
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**Theorem (Wu (’86))**

Let $\Omega \subset \mathbb{R}^{n+1}$ be domain with exterior corkscrews and suppose $\Gamma$ is $n$–AR and divides $\mathbb{R}^{n+1}$ into two NTA domains. Then $\omega_\Omega \ll \mathcal{H}^n$ on $\partial \Omega \cap \Gamma$.
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A set $\Omega \subset \mathbb{R}^{n+1}$ has **big boundary** or **n-thick** if

$$\mathcal{H}_\infty^n(B(z, r) \setminus \Omega) \geq cr^n$$

for all $z \in \partial \Omega$ and $r \in (0, \text{diam}(\partial \Omega))$.

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Identifying exterior condition - Speculations

A closed set $E \subset \mathbb{R}^{n+1}$ is called **uniformly 2-fat** or said to satisfy Capacity Density Condition **CDC** if

$$\frac{\text{Cap}(E \cap \bar{B}(z, r))}{\text{Cap}(ar{B}(z, r))} = \text{Cap}(r^{-1}(E \cap \bar{B}(z, r))) \geq c \quad \text{for all } w \in E \text{ and } r > 0.$$
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**Theorem (Lewis ('88))**

*If $E \subset \mathbb{R}^{n+1}$ is CDC then there exists some $1 < q < 2$ such that*

$$\mathcal{H}^{n+1-q}(B(w, r) \setminus \Omega) \geq cr^{n+1-q} \quad \text{for all } w \in E \text{ and } r > 0$$

*where $\Omega = \mathbb{R}^{n+1} \setminus E$. $(n + 1 - q < n)$.***
A closed set $E \subset \mathbb{R}^{n+1}$ is called uniformly 2-fat or said to satisfy Capacity Density Condition (CDC) if
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Can we replace the big boundary condition with CDC?

We[A., Badger, Bortz, Engelstein] believe that Wu’s counter example does not satisfy CDC!
Let $\Omega$ have big boundary and $\Gamma$ be ADR splits $\mathbb{R}^{n+1}$ into two NTA domains $\Omega_1, \Omega_2$. Aim: $E \subset \Gamma \cap \partial \Omega$, show $\omega^{X_0}_\Omega(E) > 0 \Rightarrow \mathcal{H}^n(E) > 0$. 

\[ \text{By David and Jerison} \] 

$\mathcal{H}^n(E) > 0$. This implies $! \\Gamma \ni \omega^{X_0}_\Omega \mathcal{H}^n$ on $\partial \Omega$.

For the sake of contradiction, suppose $\omega^{X_i}_\Gamma(E) = 0$ for all $X_i \in 1, 2$. Suffices to show that $\sup_{X_2 \in \Gamma \cap \partial \Omega} \omega^{X_0}_\Omega(E) < 1$. Then, by strong Markov property of Brownian motion, for $X_2 \in \Gamma \cap \partial \Omega$, $\omega^{X_2}_\Gamma(E) = \omega^{X_2}_\Gamma(E) + \int_{\partial \Omega \setminus \Gamma \cap \partial \Omega} \omega^{X_2}_\Gamma(E) d\omega^{X_2}_\Gamma < 0 + = < 1$.

Same holds for $X_2 \in \Gamma \cap \partial \Omega$ and hence $\sup_{X_2 \in \Gamma \cap \partial \Omega} \omega^{X_0}_\Omega(E) < 1$ which is NOT possible!
Let $\Omega$ have big boundary and $\Gamma$ be ADR splits $\mathbb{R}^{n+1}$ into two NTA domains $\Omega_1, \Omega_2$. **Aim:** $E \subset \Gamma \cap \partial \Omega$, show $\omega_{\Omega}^{X_0}(E) > 0 \Rightarrow \mathcal{H}^n(E) > 0$.

- Enough to show $\omega_{\Omega_i \cap \Omega}(E) > 0$ for some $i \in \{1, 2\}$ and $X_i \in \Omega_i$. Then by the maximum principle $\omega_{\Omega_i}(E) \geq \omega_{\Omega_i \cap \Omega}(E) > 0$. 


Sketch of the Proof

Let $\Omega$ have big boundary and $\Gamma$ be ADR splits $\mathbb{R}^{n+1}$ into two NTA domains $\Omega_1, \Omega_2$. **Aim:** $E \subset \Gamma \cap \partial \Omega$, show $\omega^X_\Omega(E) > 0 \Rightarrow H^n(E) > 0$.

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$$\omega_{\Omega_i \cap \Omega}^{X_i}(E) = 0 \text{ for all } X_i \in \Omega_i, \ i = 1, 2.$$
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Then, by strong Markov property of Brownian motion, for $X \in \Omega \cap \Omega_1$

$$\omega^X_\Omega(E) = \omega^X_{\Omega \cap \Omega_1}(E) + \int_{\partial \Omega_1 \cap \Omega} \omega^Z_\Omega(E) \, d\omega^X_{\Omega \cap \Omega_1}(Z) < 0 + \gamma = \gamma < 1.$$
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Same holds for $X \in \Omega \cap \Omega_2$ and hence

\[
\sup_{X \in \Omega} \omega^X_\Omega (E) \leq \gamma < 1 \text{ which is NOT possible!}
\]
Hence, we need to show
\[
\sup_{x \in \Gamma \cap \Omega} \omega^x_\Omega(E) \leq \gamma < 1.
\]
Hence, we need to show

\[ \sup_{x \in \Gamma \cap \Omega} \omega^X_E \leq \gamma < 1. \]

Let \( X \in \Omega \) and \( r = \text{dist}(X, \partial \Omega) \). As \( \Omega_i \) are NTA, then there are balls

\[ B^i = B(Y_i, cr) \subset \Omega_i \cap B(X, r) \text{ for } i = 1, 2. \]
Hence, we need to show
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\[ B^i = B(Y_i, cr) \subset \varOmega_i \cap B(X, r) \text{ for } i = 1, 2. \]

Geometric to show that
\[ \omega^{Y_i}_{\varOmega \cap \varOmega_i}(\Gamma \cap \varOmega) < \eta \text{ for some } \eta \in (0, 1) \text{ and } i \in \{1, 2\}. \]
Hence, we need to show
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足够的条件是证明
\[ \omega_{\Omega \cap \Omega_i}^{Y_i}(\Gamma \cap \Omega) < \eta \text{ for some } \eta \in (0, 1) \text{ and } i \in \{1, 2\}. \]

If so, the Harnack chain, and \( \omega_\Omega \) is probability measure imply that
\[
\omega_X^X(E) = 1 - \omega_X^X(E^c) \leq 1 - t \omega_{\Omega}^{Y_i}(E^c) \\
= (1 - t) + t \omega_{\Omega}^{Y_i}(E) \\
= (1 - t) + t \left( \omega_{\Omega \cap \Omega_i}^{Y_i}(E) + \int_{\partial \Omega \cap \Omega} \omega_{\Omega}^{Z}(E) \, d\omega_{\Omega \cap \Omega_1}^{Y_i}(Z) \right) \\
< (1 - t) + t(0 + \eta) = (1 - t) + t\eta =: \gamma < 1.
\]
Hence, we need to show
\[ \sup_{X \in \Gamma \cap \Omega} \omega_X^X(E) \leq \gamma < 1. \]

Let \( X \in \Omega \) and \( r = \text{dist}(X, \partial \Omega) \). As \( \Omega_i \) are NTA, then there are balls
\[ B_i^i = B(Y_i, cr) \subset \Omega_i \cap B(X, r) \text{ for } i = 1, 2. \]

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So we focus on proving \( \omega_{\Omega \cap \Omega_i}^{Y_i}(\Gamma \cap \Omega) < \eta. \)
Proof of $\omega_{Y_i}^{\Omega_1}(\Gamma \cap \Omega) < \eta$.

Let $M_0 >> 1$.

**Case 1:** There is $Z \in \partial\Omega \cap B(X, M_0 r) \cap \Omega_1$ so that $\text{dist}(Z, \Gamma) \geq \epsilon r$
Sketch of the Proof cont’

Proof of $\omega_{\Omega \cap \Omega_i}^{Y_i} (\Gamma \cap \Omega) < \eta$.

Let $M_0 >> 1$.

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In this case, Brownian motion starting at $Y^1$ has a good chance of hitting outside $\Gamma \cap \Omega$. 
Case 2: $\text{dist}(Z, \Gamma) \leq \epsilon r$ for all $Z \in \partial \Omega \cap B(X, M_0r) \cap \Omega_1$. 
Case 2: $\text{dist}(Z, \Gamma) \leq \epsilon r$ for all $Z \in \partial \Omega \cap B(X, M_0r) \cap \Omega_1$.

If black parts are $G$ then we can pick $i$ so that $\mathcal{H}^n(G) \geq \mathcal{H}^n(\partial \Omega')$. Then result of David and Jerison implies

$$1 \leq \omega_{\Omega_1}^{X_{\Omega_1}}(G) \leq \omega_{\Omega_1}^{X_{\Omega_1}}((\Gamma \cap \Omega)^c) \leq \omega_{\Omega_1}^{Y}(\Gamma \cap \Omega)^c$$

This gives $\omega_{\Omega_1}^{Y}(\Gamma \cap \Omega) < \eta$. 
Thanks!