Absolute continuity of harmonic measure on rough domains

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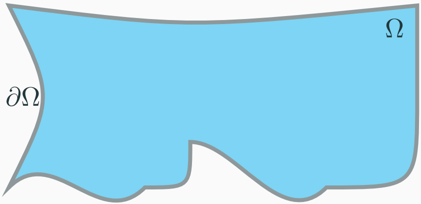
Rainwater Seminar - University of Washington

Part I

Detecting the exit point of Brownian motion

Harmonic measure

- Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, connected and open.



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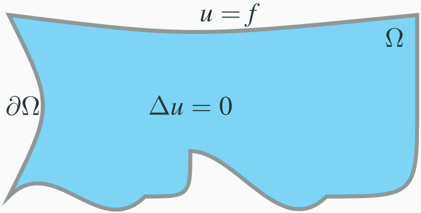

Dirichlet problem:

$$(\mathrm{D}) \quad \begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = f \text{ on } \partial\Omega \\ u \in C^2(\Omega) \cap C(\partial\Omega) \\ f \in C_c(\partial\Omega). \end{cases}$$

$$\Delta := \partial_{x_1 x_1} + \ldots + \partial_{x_{n+1} x_{n+1}}$$

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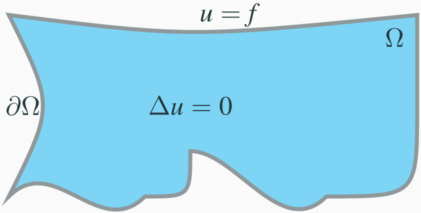

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- **Potential Theory:** $\exists!$ a family of probability measures $\{\omega_\Omega^X\}_{X \in \Omega}$ on $\partial\Omega$ called harmonic measure of Ω with a pole at $X \in \Omega$ such that

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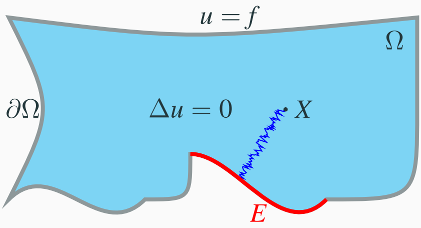

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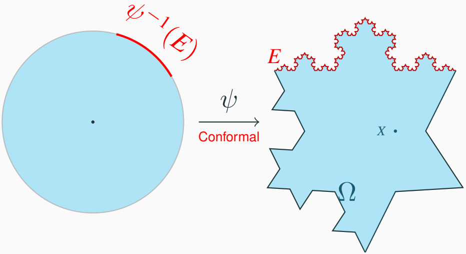
- **Probability:** Harmonic measure $\omega_\Omega^X(E)$ of E with a given pole X is the probability that a Brownian motion starting at X will first hit $\partial\Omega$ in the set E.

Examples of Harmonic Measure

- If $\Omega \subset \mathbb{R}^2$ is a simply connected domain, $\partial\Omega$ is Jordan curve



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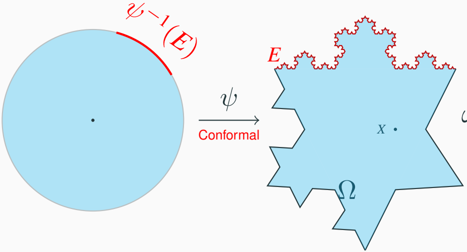

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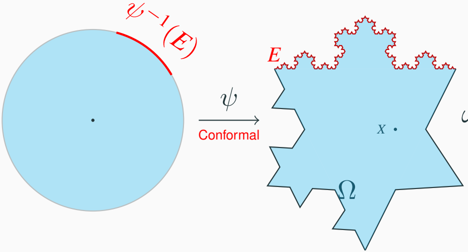

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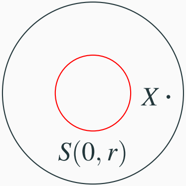
- If $\Omega = A(0, r, R) \subset \mathbb{R}^{n+1}$ is an annular region then the harmonic measure of the inner shell $S(0, r)$ is

$$\omega_\Omega^X(S(0,r)) = \begin{cases} \frac{\log R - \log |X|}{\log R - \log r} & \text{if } n = 1, \\ \frac{|X|^{2-(n+1)} - R^{2-(n+1)}}{r^{2-(n+1)} - R^{2-(n+1)}} & \text{if } n \geq 2. \end{cases}$$



More examples of harmonic Measure

Assume Ω is at least C^1 and bounded. Let $K_\Omega(X,\xi)$ be the Poisson kernel for Ω and $E \subset \partial\Omega$;

$$\omega_\Omega^X(E) = \int \chi_E(\xi) K_\Omega(X,\xi)\, d\mathcal{H}^n(\xi).$$

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$$\omega_\Omega^z(E) = \int_{\partial\Omega} \chi_{[-T,T]}(t) \frac{1}{\pi} \frac{y}{(x-t)^2+y^2} dt = \int_{-T}^{T} \frac{1}{\pi} \frac{y}{(x-t)^2+y^2} dt$$
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Notice that $\omega_\Omega^z(E)$ is a harmonic function and

$$\begin{cases} \omega_\Omega^z(E) \to 1 & \text{as } z \to E \subset \partial\Omega, \\ \omega_\Omega^z(E) \to 0 & \text{as } z \to \partial\Omega \setminus E. \end{cases}$$

Even more examples of Harmonic Measures

- If $\Omega = \mathbb{B}^{n+1}$, $(n+1)-$dimensional unit ball, and $X \in \Omega$. Then

$$\omega^X(E) = \frac{1}{\mathcal{H}^n(\mathbb{S}^n)} \int_E \frac{1-|X|^2}{|X-Y|^{n+1}} \, d\mathcal{H}^n(Y) \quad \text{for every Borel set } E \subset \mathbb{S}^n.$$

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- If $\Omega \subset \mathbb{R}^{n+1}$ is bounded domain of class C^1, then there is $K(X,Y) : \Omega \times \partial\Omega \to \mathbb{R}$ such that

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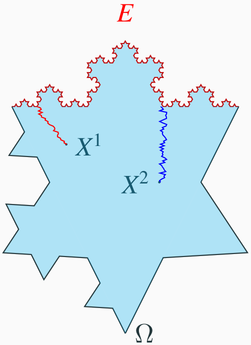
Harmonic Measure at Different Poles



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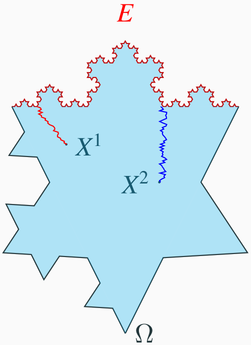

$$c^{-1}\omega^{X_1}(E) \le \omega^{X_2}(E) \le c\,\omega^{X_1}(E).$$

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- Harmonic measure ω^{X_1} and ω^{X_2} at different poles are mutually absolutely continus; $\omega^{X_1}(E) = 0 \Leftrightarrow \omega^{X_2}(E) = 0$.

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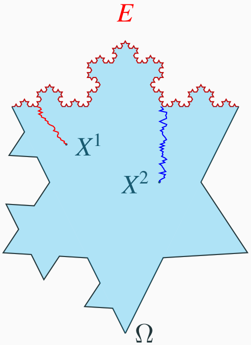

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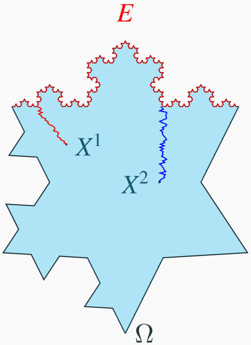

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$$c^{-1}\omega^{X_1}(E) \leq \omega^{X_2}(E) \leq c\omega^{X_1}(E).$$

- Therefore, the sets of harmonic measure zero do not depend on the pole.

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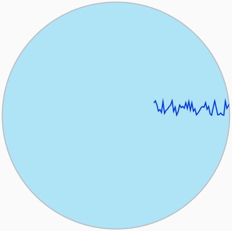
Consider a random Brownian particle moving in a domain $\Omega \subset \mathbb{R}^2$.

- Aim is to find the point where it first hits the boundary $\partial\Omega$.



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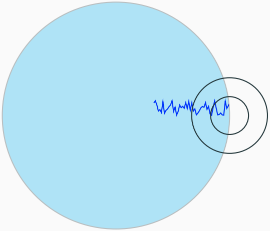

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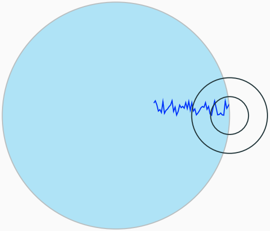

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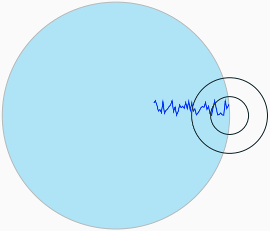

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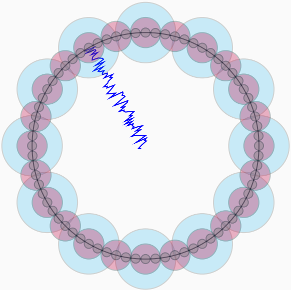
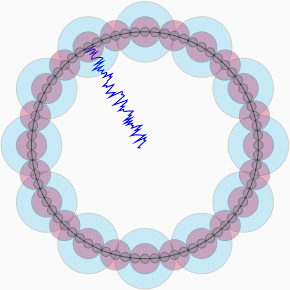


- Note that to detect an exit at x, the point must be contained in infinitely many detectors whose radii tend to zero.

Detecting the exit point of Brownian motion

- When Ω is the unit disk $\mathbb{D}$, and the Brownian particle starts at 0 then the hitting distribution on $\partial\Omega$ is normalized Lebesgue measure.
- Thus to detect the exit point almost surely, we must cover almost every point of $\partial\Omega$ by arbitrarily small balls.



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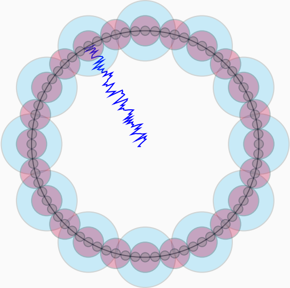
- If $\phi(r) \geq r$ then we can not detect the exit point on a finite budget.
- However, if $\phi(r) = o(r)$ then we can cover $\partial\Omega$ by about n_k balls of size $1/n_k$ and let $n_k \nearrow \infty$ so fast that $\sum n_k \phi(1/n_k) < \infty$.



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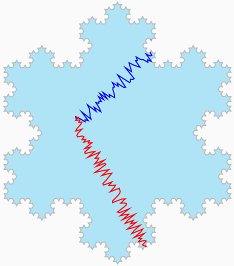

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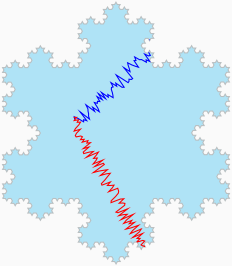

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- However, not all parts of the snowflake are equally likely to be hit by Brownian motion, and there is a small subset of $\partial\Omega$ which still gets hit with probability 1.

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$\mathcal{H}^n_\infty(E)$ is called the Hausdorff content of E and is defined as

$$\mathcal{H}^n_\infty(E) = \inf \left\{ \sum_{i=1}^{\infty} (r_i)^n;\ E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i) \right\}.$$

- $\mathcal{H}^\alpha_\infty(E) \leq \mathcal{H}^\alpha_\delta(E) \leq \mathcal{H}^\alpha(E)$. But still $\mathcal{H}^\alpha_\infty(E) = 0 \iff \mathcal{H}^\alpha(E) = 0$.

Being singular $\perp$ – absolutely continuous $\ll$

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The dimension of a measure μ is the smallest dimension of a full μ-measure set, i.e.,

$$\dim_{\mathcal{H}}(\mu) = \inf\{\dim_{\mathcal{H}}(E) :\ \mu(E^c) = 0\} = \inf\{\alpha :\ \mu \perp \mathcal{H}^\alpha\}$$

- $\mu \perp \nu$ if there is a set E such that $\mu(E) = \nu(E^c) = 0$
- $\mu \ll \nu$ if $\nu(E) = 0 \Rightarrow \mu(E) = 0$.
- $\mu \sim \nu$ if $\nu \ll \mu \ll \nu$.

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- $\mu \ll \nu$ if $\nu(E) = 0 \Rightarrow \mu(E) = 0$.
- $\mu \sim \nu$ if $\nu \ll \mu \ll \nu$.

It is always true that $\dim_{\mathcal{H}}(\mu) \leq \dim_{\mathcal{H}}(\mathrm{supp}(\mu))$.

Being singular $\perp$ – absolutely continuous $\ll$

The Hausdorff dimension of a set E is defined by

$$\dim_{\mathcal{H}}(E) = \inf\{\alpha;\ \mathcal{H}^{\alpha}(E) = 0\}.$$

The smaller α is, the more expensive it is to cover E; the dimension marks the transition from positive to zero cost coverings.

The dimension of a measure μ is the smallest dimension of a full μ-measure set, i.e.,

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Thus the detection question is really:

- For which ϕ we have $\omega \perp \mathcal{H}_{\phi}$ and when is $\omega \ll \mathcal{H}_{\phi}$?

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Theorem (F. and M. Riesz(1916))

Let Ω be a simply connected domain in the plane with $\mathcal{H}^1(\partial\Omega) < \infty$. Let $\psi : \mathbb{D} \to \Omega$ be conformal.

Then $\psi' \in L^1(\partial\mathbb{D})$. Moreover, for any Borel set $E \subset \partial\mathbb{D}$,

$$\mathcal{H}^1(\psi(E)) = \int_E |\psi'(e^{\mathbf{i}\theta})|\, d\theta.$$

Hence, using $\omega_\Omega^z(K) = 1/2\pi$ arclength$(\psi^{-1}(K))$, $K \subset \partial\Omega$, one has

$$\omega(A) = 0 \quad \Longleftrightarrow \quad \mathcal{H}^1(A) = 0 \quad \textit{whenever } A \subset \partial\Omega \textit{ Borel.}$$

$$\textit{i.e.} \quad \omega \ll \mathcal{H}^1 \ll \omega \quad \textit{on } \partial\Omega.$$

Thanks!

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We can detect a.e. exit point of Brownian motion on a finite ϕ-budget iff there is a Vitali covering of a full ω-measure set E by balls of radius $\{r_j\}$ such that $\sum \phi(r_j) < \infty$.

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Part II

Absolute continuity of harmonic measure on rough domains

Necessary and sufficient conditions for absolute continuity

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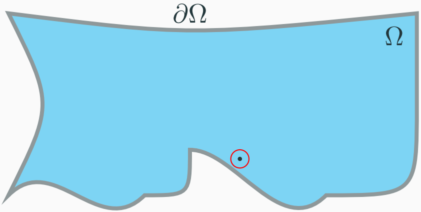
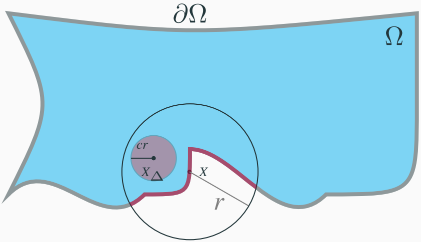
Non-tangentially Accessible Domains(NTA)

- Openness



Non-tangentially Accessible Domains(NTA)

- Openness $\rightsquigarrow$ Corkscrew condition.



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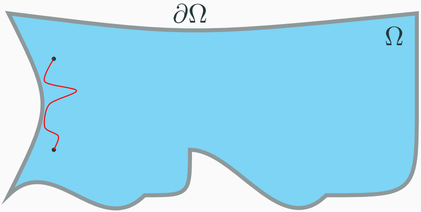
- Path-connectedness



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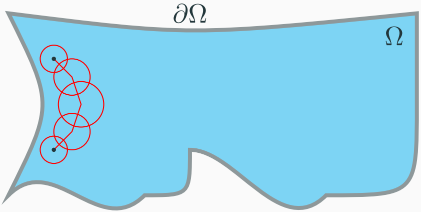

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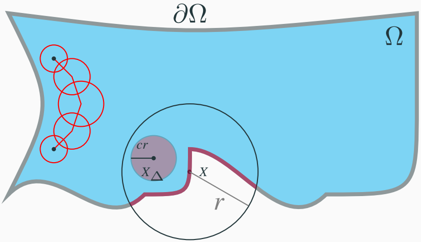

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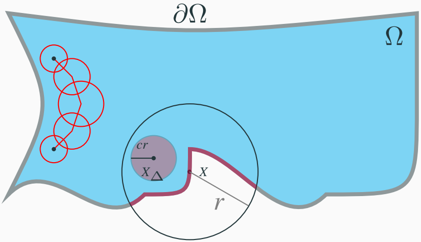

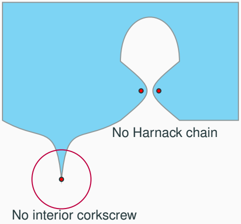

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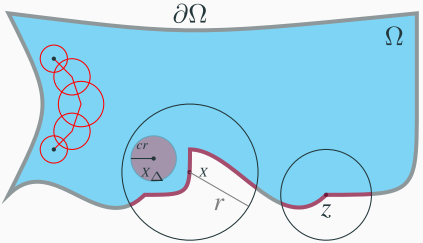
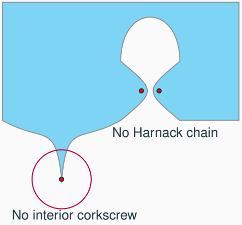


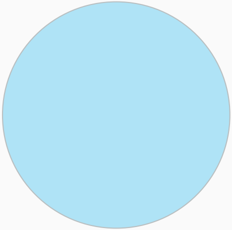


- Ω is NTA $\equiv \begin{cases} \text{• **Interior** Corkscrew and Harnack Chain.} \\ \text{• **Exterior** Corkscrew.} \end{cases}$
- $\partial\Omega$ is $n-$Ahlfors regular (AR) if

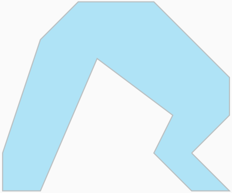
$$cr^n \leq \mathcal{H}^n(\partial\Omega \cap B(z,r)) \leq cr^n \text{ whenever } z \in \partial\Omega \text{ and } r \in (0, \operatorname{diam}(\partial\Omega)).$$

Examples of such domains

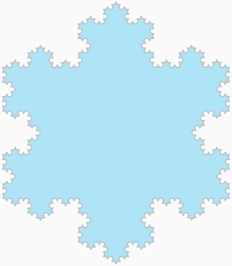
Smooth Domains



Lipschitz Domains



NTA Domains



- NTA domains need not be graph domains or of finite perimeter.

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Let $\Sigma = f(\mathbb{R}^n)$ be a Lipschitz image of $\mathbb{R}^n$.

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i.e. $$E \subset \left(\bigcup_{i=1}^{\infty} \Sigma_i \right) \cup \Sigma_0 \qquad \text{with } \mathcal{H}^n(\Sigma_0) = 0.$$

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- [Besicovitch-Federer] E is $n-$purely unrectifiable if $0 < \mathcal{H}^n(E) < \infty$ and $\mathcal{H}^n(\pi_L(E)) = 0$ for almost every $n-$dimensional plane $L \subset \mathbb{R}^{n+1}$.

Here π_L denotes the orthogonal projection of $\mathbb{R}^{n+1}$ onto L.

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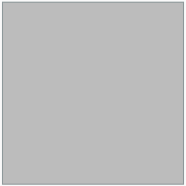

$\mathcal{C}_0$

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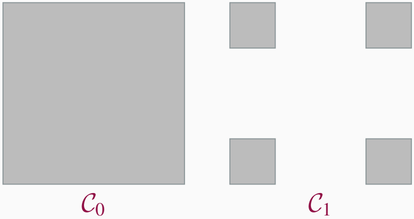

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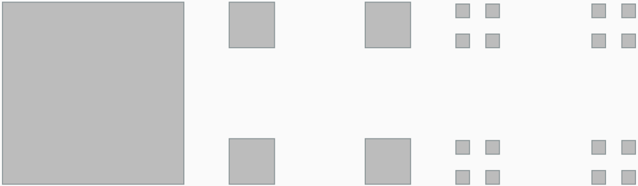

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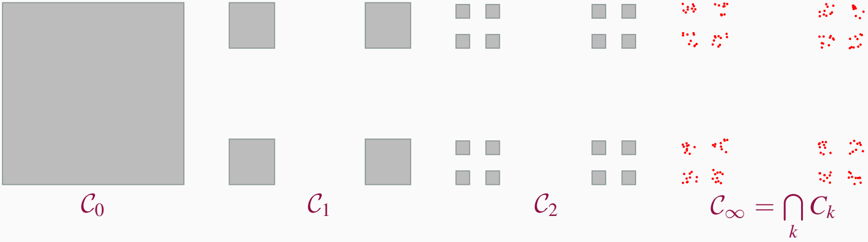

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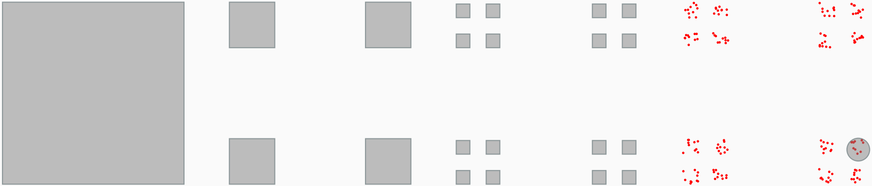

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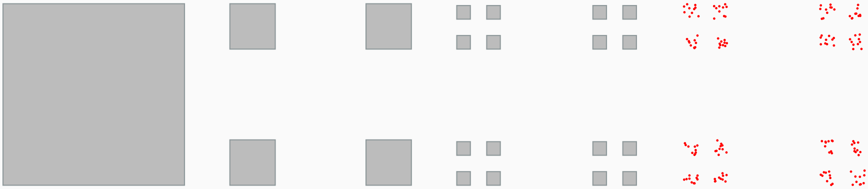

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  - For almost every line L in $\mathbb{R}^2$, $\mathcal{H}^1(\pi_L(\mathcal{C}_\infty)) = 0$.
- Hence $\mathcal{C}_\infty$ is a purely 1-unrectifiable.
- Every rectifiable curve intersects $\mathcal{C}_\infty$ in a set of zero $\mathcal{H}^1$-measure.

Global results in higher dimension

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- **David and Jerison**('90);
- **Semmes**('89):

If Ω is NTA and $\partial\Omega$ is AR then $\omega \sim \mathcal{H}^n$ on $\partial\Omega$.

- **Badger**('12): If $\Omega \subset \mathbb{R}^{n+1}$ is NTA with $\mathcal{H}^n(\partial\Omega) < \infty$ then $\omega \ll \mathcal{H}^n$ on

$$\left\{ x \in \partial\Omega;\ \liminf_{r\to 0} \frac{\mathcal{H}^n(\partial\Omega \cap B(x,r))}{r^n} < \infty \right\}.$$

- **Azzam, Mourgoglou, and Tolsa**('15): $\exists$ NTA domain Ω with $\mathcal{H}^n(\partial\Omega) < \infty$ such that $\omega \not\ll \mathcal{H}^n|_{\partial\Omega}$ (Using the deep result of Wolff which was further developed by Lewis, Nyström, Vogel).

Necessary conditions for Absolute Continuity

- **Pommerenke**('86): If $\Omega \subset \mathbb{C}$ is simply connected and $\omega \ll \mathcal{H}^1$ on a set $E \subset \partial\Omega$ then ω a.e. point E is a cone point for Ω and $\omega-$almost all of F can be covered by a countable union of $1-$dimensional (possibly rotated) Lipschitz graphs (i.e., $\omega|_E$ is rectifiable).

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Theorem (Azzam-Hofmann-Martell-Mayboroda-Mourgoglou-Tolsa-Volberg, ('15))

- *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$,* **open** *and* **connected**.
- *Let $F \subset \partial\Omega$ with $\mathcal{H}^n(F) < \infty$.*

1. *If $\omega \ll \mathcal{H}^n$ on $F \implies \omega|_F$ is $n-$rectifiable.*
2. *If $\mathcal{H}^n \ll \omega$ on $F \implies F$ is $n-$rectifiable.*

★ Portion of the boundary should be contained in a nice rectifiable set(like a graph or curve)!

Notion of Capacity - First Definition

Let E be a closed subset of $\mathbb{R}^{n+1}$, $n \geq 2$. Then

$$\text{Cap}(E) = \inf \left\{ \int |\nabla v|^2 dx,\ v \in C_0^\infty(\mathbb{R}^{n+1}),\ v \geq 1 \text{ on } E \right\}.$$

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If u is the minimizer of energy then u weakly satisfies

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^{n+1} \setminus E, \\ u = 1 & \text{on E}, \\ u \to 0 & \text{as } |x| \to \infty. \end{cases}$$

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Then,

$$\text{Cap}(E) = \gamma = \lim_{|x| \to \infty} \frac{u(x)}{|x|^{(n+1)-2}}.$$

This definition is called the electrostatic capacity of E.

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Then Cap$(E) = \nu(E)$ for a measure ν which is called equilibrium measure and satisfying $U_\nu(x) \leq 1$ for $x \in$ supp(ν) and $U_\nu(x) \geq 1$ up to a set of measure zero capacity on E. Note that U_ν is a positive super harmonic function in $\mathbb{R}^{n+1}$ and harmonic outside of E.

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$$\mathrm{Cap}(E) = [\inf\{\mathcal{K}_\mu :\ \mu(E) = 1,\ \mu(E^c) = 0\}]^{-1}$$

where

$$\mathcal{K}_\mu = \iint_{E\times E} \frac{1}{|x-y|^{(n+1)-2}} d\mu(x) d\mu(y).$$

which denotes the energy of μ with respect to the kernel $1/|x|^{(n+1)-2}$.

Counter example of Wu revisited.

In $\mathbb{R}^2$, there exists simply connected Jordan domain K satisfying

(1) $K \cap \{x : x_1 > 0\} \subset \{x : |x| < 2\}$, $K \cap \{x : x_1 < 0\} = \{x : x_1 < 0, |x| < 3\}$

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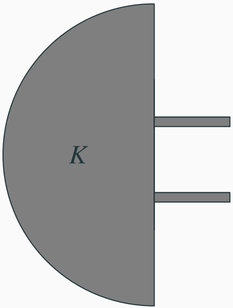

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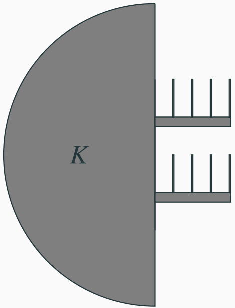

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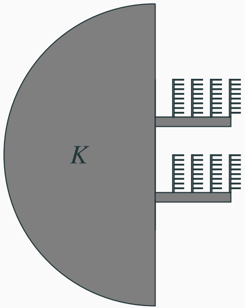

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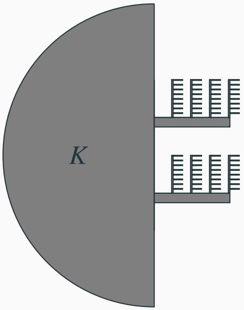

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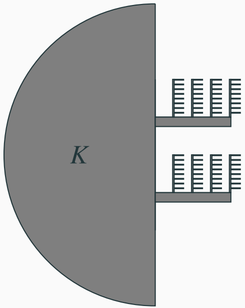

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Key point here is that for $0 < \eta < \epsilon$,
$\mathrm{Cap}_3(K_\epsilon \setminus \bar{K}_\eta) < \frac{1}{100}\mathrm{Cap}_3(\partial_2 K)$

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A set $\Omega \subset \mathbb{R}^{n+1}$ has **big boundary** or **n-thick** if

$$\mathcal{H}^n_\infty(B(z,r) \setminus \Omega) \geq cr^n \quad \text{for all } z \in \partial\Omega \text{ and } r \in (0, \operatorname{diam}(\partial\Omega)).$$

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Theorem (A., Azzam, Mourgoglou ('16))

Suppose $\Omega \subset \mathbb{R}^{n+1}$ has big boundary and let $\Gamma \subset \mathbb{R}^{n+1}$ is $n-$AR and splits $\mathbb{R}^{n+1}$ into two NTA domains. Then $\omega_\Omega \ll \mathcal{H}^n$ on $\partial\Omega \cap \Gamma$.

Identifying exterior condition - Speculations

A closed set $E \subset \mathbb{R}^{n+1}$ is called **uniformly 2-fat** or said to satisfy Capacity Density Condition **CDC** if

$$\frac{\mathrm{Cap}(E \cap \bar{B}(z,r))}{\mathrm{Cap}(\bar{B}(z,r)} = \mathrm{Cap}(r^{-1}(E \cap \bar{B}(z,r)) \geq c \quad \text{for all } w \in E \text{ and } r > 0.$$

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If $E \subset \mathbb{R}^{n+1}$ is CDC then there exists some $1 < q < 2$ such that

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We[A., Badger, Bortz, Engelstein] believe that Wu's counter example does not satisfy CDC!

Sketch of the Proof

Let Ω have big boundary and Γ be ADR splits $\mathbb{R}^{n+1}$ into two NTA domains Ω_1, Ω_2. Aim: $E \subset \Gamma \cap \partial\Omega$, show $\omega_\Omega^{X_0}(E) > 0 \Rightarrow \mathcal{H}^n(E) > 0$.

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Then, by strong Markov property of Brownian motion, for $X \in \Omega \cap \Omega_1$

$$\omega_\Omega^X(E) = \omega_{\Omega \cap \Omega_1}^X(E) + \int_{\partial\Omega_1 \cap \Omega} \omega_\Omega^Z(E) \, d\omega_{\Omega \cap \Omega_1}^X(Z) < 0 + \gamma = \gamma < 1.$$

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Let Ω have big boundary and Γ be ADR splits $\mathbb{R}^{n+1}$ into two NTA domains Ω_1, Ω_2. Aim: $E \subset \Gamma \cap \partial\Omega$, show $\omega_\Omega^{X_0}(E) > 0 \Rightarrow \mathcal{H}^n(E) > 0$.

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- By David and Jerison $\mathcal{H}^n(E) > 0$. This implies $\omega_\Omega \ll \mathcal{H}^n$ on $\partial\Omega$.
- For the sake of contradiction, suppose

$$\omega_{\Omega_i \cap \Omega}^{X_i}(E) = 0 \text{ for all } X_i \in \Omega_i,\ \ i = 1, 2.$$

- Suffices to show that $\sup\limits_{X \in \Gamma \cap \Omega} \omega_\Omega^X(E) \leq \gamma < 1$.

Then, by strong Markov property of Brownian motion, for $X \in \Omega \cap \Omega_1$

$$\omega_\Omega^X(E) = \omega_{\Omega \cap \Omega_1}^X(E) + \int_{\partial\Omega_1 \cap \Omega} \omega_\Omega^Z(E)\, d\omega_{\Omega \cap \Omega_1}^X(Z) < 0 + \gamma = \gamma < 1.$$

Same holds for $X \in \Omega \cap \Omega_2$ and hence

$$\sup_{X \in \Omega} \omega_\Omega^X(E) \leq \gamma < 1 \text{ which is NOT possible!}$$

Sketch of the Proof cont'

Hence, we need to show

$$\sup_{X \in \Gamma \cap \Omega} \omega_{\Omega}^{X}(E) \leq \gamma < 1.$$

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Let $X \in \Omega$ and $r = \text{dist}(X, \partial\Omega)$. As Ω_i are NTA, then there are balls

$$B^i = B(Y_i, cr) \subset \Omega_i \cap B(X, r) \text{ for } i = 1, 2.$$

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- Enough to show that

$$\omega_{\Omega \cap \Omega_i}^{Y_i}(\Gamma \cap \Omega) < \eta \text{ for some } \eta \in (0,1) \text{ and } i \in \{1, 2\}.$$

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If so, the Harnack chain, and ω_Ω is probability measure imply that

$$\begin{aligned}
\omega_\Omega^X(E) &= 1 - \omega_\Omega^X(E^c) \leq 1 - t\omega_\Omega^{Y_i}(E^c) \\
&= (1 - t) + t\omega_\Omega^{Y_i}(E) \\
&= (1 - t) + t\left(\omega_{\Omega \cap \Omega_i}^{Y_i}(E) + \int_{\partial\Omega_i \cap \Omega} \omega_\Omega^Z(E)\, d\omega_{\Omega \cap \Omega_1}^{Y_i}(Z)\right) \\
&< (1 - t) + t(0 + \eta) = (1 - t) + t\eta =: \gamma < 1.
\end{aligned}$$

Sketch of the Proof cont'

Hence, we need to show

$$\sup_{X\in\Gamma\cap\Omega} \omega_\Omega^X(E) \leq \gamma < 1.$$

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\end{aligned}$$

So we focus on proving $\omega_{\Omega\cap\Omega_i}^{Y_i}(\Gamma\cap\Omega) < \eta$.

Sketch of the Proof cont'

Proof of $\omega^{Y_i}_{\Omega\cap\Omega_i}(\Gamma\cap\Omega) < \eta$.

Let $M_0 >> 1$.

Case 1: There is $Z \in \partial\Omega \cap B(X, M_0 r) \cap \Omega_1$ so that $\mathrm{dist}(Z, \Gamma) \geq \epsilon r$

Sketch of the Proof cont'

Proof of $\omega^{Y_i}_{\Omega\cap\Omega_i}(\Gamma\cap\Omega) < \eta$.

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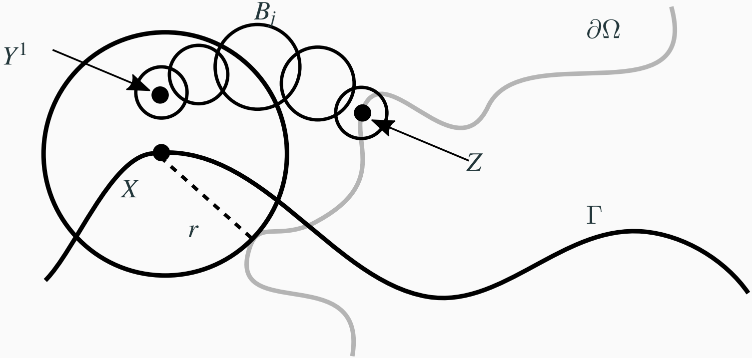
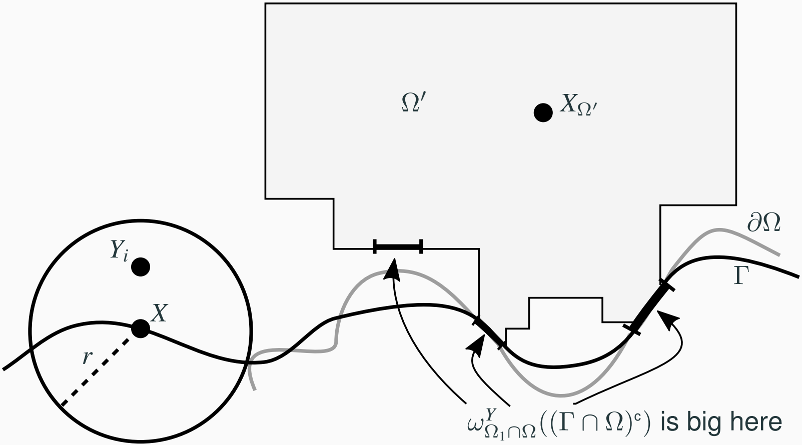


In this case, Brownian motion starting at Y^1 has a good chance of hitting outside $\Gamma \cap \Omega$.

Case 2: dist$(Z,\Gamma) \le \epsilon r$ for all $Z \in \partial\Omega \cap B(X, M_0 r) \cap \Omega_1$.



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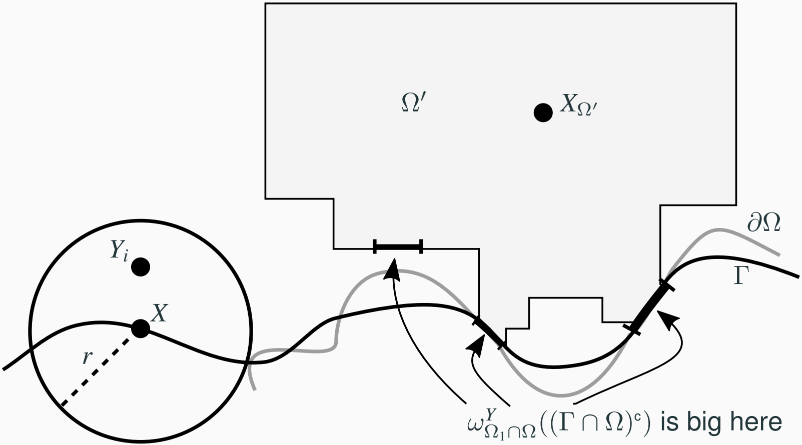

If black parts are G then we can pick i so that $\mathcal{H}^n(G) \geq \mathcal{H}^n(\partial\Omega')$.
Then result of David and Jerison implies

$$1 \lesssim \omega^{X_{\Omega'}}_{\Omega'}(G) \lesssim \omega^{X_{\Omega'}}_{\Omega_1\cap\Omega}((\Gamma\cap\Omega)^{\mathsf{c}}) \lesssim \omega^{Y}_{\Omega_1\cap\Omega}((\Gamma\cap\Omega)^{\mathsf{c}})$$

This gives $\omega^Y_{\Omega_1\cap\Omega}(\Gamma\cap\Omega) < \eta$.

Thanks!