Rectifiability, interior approximation, absolute continuity of harmonic measure

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Joint with M. Badger, S. Bortz, S. Hofmann, J. M. Martell

$\mathcal{H}^n$: $n-$dimensional Hausdorff measure and Rectifiability in $\mathbb{R}^{n+1}$

Let $A \subset \mathbb{R}^{n+1}$, $0 \leq n < \infty$, $0 < \delta \leq \infty$.

$$\mathcal{H}^n_\delta(A) = \inf\{\sum(\mathrm{diam}(E_i))^n;\ A \subset \bigcup_{i=1}^{\infty} E_i,\ \mathrm{diam}(E_i) \leq \delta\}.$$

$$\mathcal{H}^n(A) := \lim_{\delta \to 0} \mathcal{H}^n_\delta(A) = \sup_{\delta > 0} \mathcal{H}^n_\delta(A).$$

- $E \subset \mathbb{R}^{n+1}$ is $n-$rectifiable if there exists a family $\{\Sigma_i\}_i$ of Lipschitz images of $\mathbb{R}^n$ such that

$$\mathcal{H}^n\left(E \setminus \bigcup_{i=1}^{\infty} \Sigma_i\right) = 0,$$

- $E \subset \mathbb{R}^{n+1}$ is $n-$purely unrectifiable if E contains NO $n-$rectifiable set F with $\mathcal{H}^n(F) > 0$.

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Rectifiability and Absolute Continuity

Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain and let ω be harmonic measure for Ω.

Question

1. Under what conditions, one has $\omega \ll \mathcal{H}^n$ and/or $\mathcal{H}^n \ll \omega$ on $\partial\Omega$?
2. What are the implications of $\omega \ll \mathcal{H}^n$ and $\mathcal{H}^n \ll \omega$ on $\partial\Omega$?

- **F. and M. Riesz**(1916): If $\Omega \subset \mathbb{R}^2$ is simply connected, $\mathcal{H}^1(\partial\Omega) < \infty$ then

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- **Lavrentiev**(1936): Quantitative version.
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Non-Tangentially Accessible Domains & Ahlfors-David Regularity

- Ω is NTA $\equiv$ $\begin{cases} \bullet\ \textbf{Interior}\ \text{Corkscrew and Harnack Chain.} \\ \bullet\ \textbf{Exterior}\ \text{Corkscrew.} \end{cases}$
- Ω is 1-sided NTA $\equiv$ **Interior** Corkscrew and Harnack Chain.

Credit: Chema Martell

- NTA domains need not be graph domains or of finite perimeter.
- E is called $n-$**Ahlfors-David regular (ADR)** if

$$cr^n \leq \mathcal{H}^n(E \cap B(x,r)) \leq cr^n \text{ whenever } x \in \partial\Omega.$$

- **ADR** = **Lower** ADR + **Upper** ADR.

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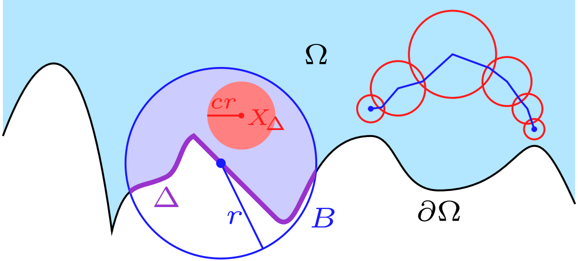

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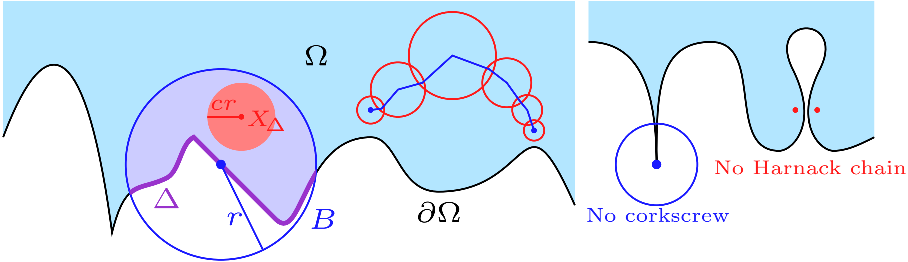

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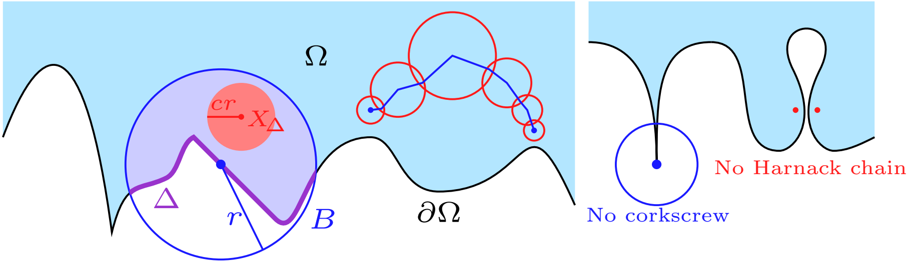

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A_∞ and A_∞^{weak} conditions

Let $E \subset \mathbb{R}^{n+1}$ be ADR set and let $\Delta_0 = E \cap B(z,r)$, $z \in E$.

A_∞ Condition

$\omega \in A_\infty(\mathcal{H}^n|_{\Delta_0})$ if there exist C and θ such that for all $\Delta = B(x,r') \cap E$ where $x, \in E$ and $B(x,r') \subset B(z,r)$ one has

$$\frac{\omega(F)}{\omega(\Delta)} \le C \left(\frac{\mathcal{H}^n(F)}{\mathcal{H}^n(\Delta)} \right)^\theta \text{ for every Borel set } F \subset \Delta.$$

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Global results in higher dimension

- **Dahlberg**(1977): Ω is a **Lipschitz** domain then $\omega \in A_\infty(\mathcal{H}^n|_{\partial\Omega})$.
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- **Badger**(2012):
  1. If $\Omega \subset \mathbb{R}^{n+1}$ is **NTA** then $\omega \ll \mathcal{H}^n \ll \omega$ on a $n-$rectifiable set $A \subset \partial\Omega$

  $$\mathbf{A} = \left\{ x \in \partial\Omega;\ \liminf_{r\to 0} \frac{\mathcal{H}^n(\partial\Omega \cap B(x,r))}{r^n} < \infty \right\}.$$

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★ Portions of the **boundary** should be contained in a **nice** set(like a graph or curve).

- **Azzam, Mourgoglou, and Tolsa**(2015): $\exists$ NTA domain Ω with $\mathcal{H}^n(\partial\Omega) < \infty$ such that $\omega \not\ll \mathcal{H}^n|_{\partial\Omega}$.

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A characterization of uniform rectifiablity

Theorem

Let Ω be **1-sided NTA** and $\partial\Omega$ **ADR**. TFAE

1. Ω is NTA domain (and therefore it is chord-arc domain).
2. $w \in A_\infty(\mathcal{H}^n|_{\partial\Omega})$.
3. $w \in A_\infty^{\text{weak}}(\mathcal{H}^n|_{\partial\Omega})$.
4. $\partial\Omega$ is Uniformly Rectifiable[=ADR+Big Pieces of Lipschitz Images]

1 $\Longrightarrow$ 2 by David and Jerison and independently by Semmes.

2 $\Longrightarrow$ 3 is trivial.

3 $\Longrightarrow$ 4 by Hofmann, Martell, and Uriarte-Tuero.

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4 $\implies$ 1 by Azzam, Hofmann, Martell, Nyström, and Toro.

Theorem A (A., Badger, Hofmann, Martell)

Let Ω be **1-sided NTA** *and $\partial\Omega$ be* **ADR**. *TFAE;*

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Is Connectivity really required?

All results requires some strong connectivity hypothesis;

1 **Simply Connected** or 2 **Harnack Chain** or 3 **Corkscrew**

Theorem[$\Rightarrow$ Bortz and Hofmann, $\Leftarrow$ Hofmann and Martell]

Let E be **ADR** and let $\Omega = \mathbb{R}^{n+1} \setminus E$. Then

$$E \text{ is Uniformly Rectifiable} \iff E \text{ has BPGHME.}$$

BPGHME= Big Pieces of Good Harmonic Measure Estimates:

- A $Q \in \mathbb{D}(E)$ then $\exists \Omega_Q \subset \Omega$.
- B $\partial\Omega_Q$ is ADR.
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- D $\partial\Omega$ and $\partial\Omega_Q$ have a big overlap; $\mathcal{H}^n(\partial\Omega \cap Q) \gtrsim \mathcal{H}^n(Q)$.
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Absolute continuity implies rectifiability

Theorem[AHM^{3}TV]

- Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, **open** and **connected**.
- Let $F \subset \partial\Omega$ with $\mathcal{H}^n(F) < \infty$.

1. If $\omega_\Omega \ll \mathcal{H}^n$ on $F \implies \omega_\Omega|_F$ is $n-$rectifiable.
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A Radon measure μ on $\mathbb{R}^{n+1}$ is n-rectifiable if its (any) Borel support can be covered by countably many (rotated) graphs of scalar Lipschitz functions on $\mathbb{R}^n$ up to zero $\mu-$measure.

[AHM^{3}TV]=Azzam, Hofmann, Martell, Mayboroda, Mourgoglou, Tolsa, and Volberg.

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Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain and let ω be the harmonic measure for Ω.

Question

1: Under what conditions, we have

$$\text{A} \quad \omega \ll \mathcal{H}^n \qquad \text{and/or} \qquad \text{B} \quad \mathcal{H}^n \ll \omega \ \text{ on } \partial\Omega?$$

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Weakening the Lower Ahlfors-David Regularity condition

- $E \subset \mathbb{R}^{n+1}$, $n \geq 1$, closed set with locally finite $\mathcal{H}^n$-measure.

Lower Ahlfors-David Regularity: $r^n \lesssim \mathcal{H}^n(E \cap B(x,r))$, $\forall x \in E$.

WLADR

$\mathcal{H}^n|_E$ satisfies the **Weak Lower Ahlfors-David regular condition (WLADR)** if

$$\mathcal{H}^n(E \setminus E_*) = 0,$$

$$E_* = \left\{ x \in E : \inf_{\substack{\mathbf{y} \in \mathbf{B}(\mathbf{x},\rho) \cap \mathbf{E} \\ \mathbf{0} < \mathbf{r} < \rho}} \frac{\mathcal{H}^n(B(y,r) \cap E)}{r^n} > 0, \text{ for some } \rho > 0 \right\}.$$

i.e.: For $x \in E_*$, there exists a small ball B_x center at x and a constant c_x such that the lower ADR condition holds for all balls $B \subset B_x$ with constant c_x.

- WLADR is **weaker** than Lower ADR.

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Weakening the interior corkscrew condition

Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain, $n \geq 1$.

Interior corkscrew condition: for some uniform constant c, $0 < c < 1$, and for every ball $B(x,r)$ centered on $\partial\Omega$ with $0 < r < \operatorname{diam}(\partial\Omega)$, there is a ball $B(\tilde{x}, cr) \subset B(x,r) \cap \Omega$.

Interior Measure Theoretic Boundary

The **Interior Measure Theoretic Boundary** $\partial_+\Omega$ is defined as

$$\partial_+\Omega := \left\{ x \in \partial\Omega : \limsup_{r \to 0^+} \frac{|B(x,r) \cap \Omega|}{|B(x,r)|} > 0 \right\}.$$

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Covering of E with boundaries of bounded Lipschitz domains

Theorem B (A., Bortz, Hofmann, Martell)

- *Let* $E \subset \mathbb{R}^{n+1}$, $n \geq 1$, *be a closed set,*
- E *have locally finite* $\mathcal{H}^n$*-measure,*
- E *satisfy the WLADR condition.*

Then

$$E \text{ is } n\text{-rectifiable} \iff E \subset Z \cup \Big(\bigcup_j \partial\Omega_j\Big).$$

(A) $\{\Omega_j\}_j$ *is a countable collection of bounded Lipschitz domains,*

(B) $\Omega_j \subset \mathbb{R}^{n+1} \setminus E$ *for every* j*,*

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Covering of E with boundaries of bounded Lipschitz domains

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Rectifiability is necessary and sufficient for absolute continuity

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Mourgoglou: Under stronger assumption that the reduced boundary $\partial^*\Omega$ agrees with $\partial\Omega$ $\mathcal{H}^n$ almost everywhere.

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Local Result

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