Rectifiability, interior approximation and Absolute continuity of Harmonic Measure

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$\mathcal{H}^n$: $n-$dimensional Hausdorff measure in $\mathbb{R}^m$

Let $A \subset \mathbb{R}^m$, $0 \le n < \infty$, $0 < \delta \le \infty$.

$$\mathcal{H}^n_\delta(A) = \inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam}(E_i))^n;\ A \subset \bigcup_{i=1}^{\infty} E_i,\ \operatorname{diam}(E_i) \le \delta\right\}.$$

- $\mathcal{H}^n(A) = \lim_{\delta \to 0} \mathcal{H}^n_\delta(A) = \sup_{\delta > 0} \mathcal{H}^n_\delta(A)$.

Properties of $\mathcal{H}^n$.

1. $\mathcal{H}^n$ is a Borel measure.
2. Translation invariant: $\mathcal{H}^n(\lambda E) = \lambda^n \mathcal{H}^n(E)$ for all $\lambda > 0$.
3. $\mathcal{H}^s \equiv 0$ for $s > m$.
4. If $\alpha > \alpha'$ then $\mathcal{H}^\alpha(E) > 0 \to \mathcal{H}^{\alpha'}(E) = \infty$.
5. If $f : \mathbb{R}^m \to \mathbb{R}^s$ is a Lipschitz then $\mathcal{H}^n(f(E)) \le \operatorname{Lip}(f)^n \mathcal{H}^n(E)$.
6. $\mathcal{H}^m$ measure coincides with the Lebesgue measure.

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Rectifiability of a set

Let $\Sigma = f(\mathbb{R}^n)$ be a Lipschitz image of $\mathbb{R}^n$.

- $E \subset \mathbb{R}^m$ is $n-$rectifiable, $n \in \{1, \ldots, m\}$, if there exists a family $\{\Sigma_i\}_i$ of Lipschitz images of $\mathbb{R}^n$ such that

$$\mathcal{H}^n\left(E \setminus \bigcup_{i=1}^{\infty} \Sigma_i\right) = 0,$$

i.e. $$E \subset \left(\bigcup_{i=1}^{\infty} \Sigma_i\right) \cup \Sigma_0 \qquad \text{with } \mathcal{H}^n(\Sigma_0) = 0.$$

- $E \subset \mathbb{R}^m$ is $n-$purely unrectifiable if $0 < \mathcal{H}^n(E) < \infty$ and $\mathcal{H}^n(\pi_L(E)) = 0$ for almost every $n-$dimensional plane $L \subset \mathbb{R}^m$.

- ⋆ $E \subset \mathbb{R}^m$ is $n-$purely unrectifiable if E contains NO $n-$rectifiable set F with $\mathcal{H}^n(F) > 0$.

Here π_L denotes the orthogonal projection of $\mathbb{R}^m$ onto L.

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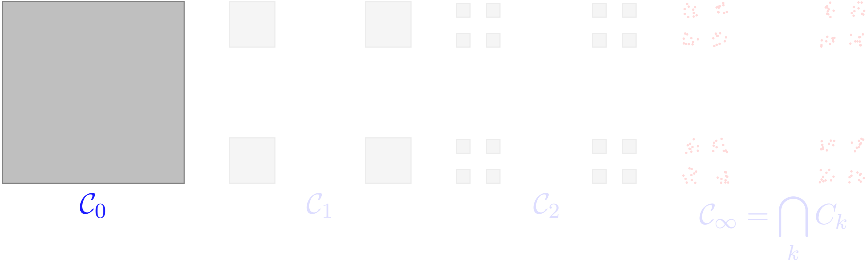
An example of a purely unrectifiable set

The usual example is 4-corner Cantor set.



- There exists $c > 1$ such that for each $z \in \mathcal{C}_\infty$ and $r \in (0, \sqrt{2})$

$$c^{-1} r \leq \mathcal{H}^1(\mathcal{C}_\infty \cap B(z,r)) \leq cr$$

- For almost every line L in $\mathbb{R}^2$, $\mathcal{H}^1(\pi_L(\mathcal{C}_\infty)) = 0$.
- Hence $\mathcal{C}_\infty$ is a purely 1-unrectifiable.

- Every rectifiable curve intersects $\mathcal{C}_\infty$ in a set of zero $\mathcal{H}^1$-measure.

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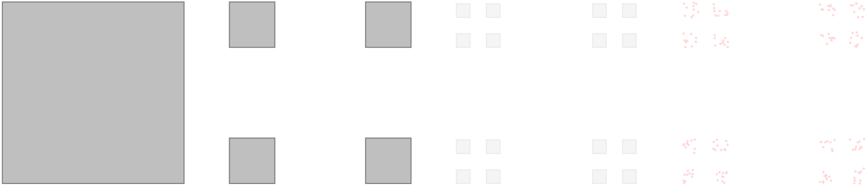

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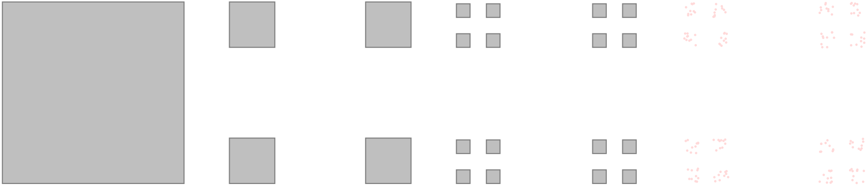

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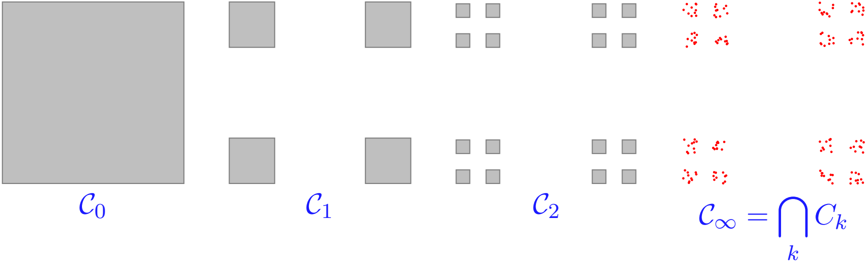

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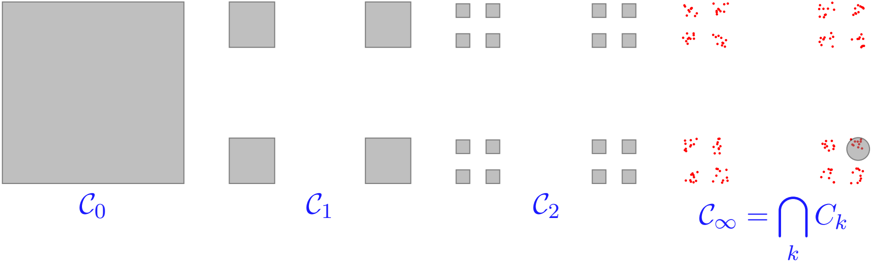

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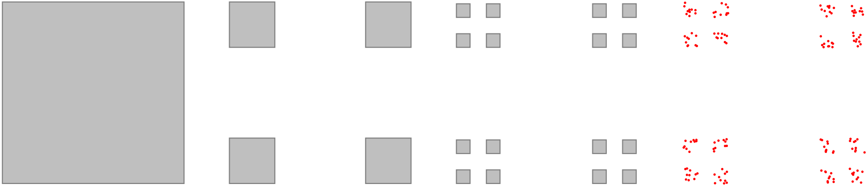

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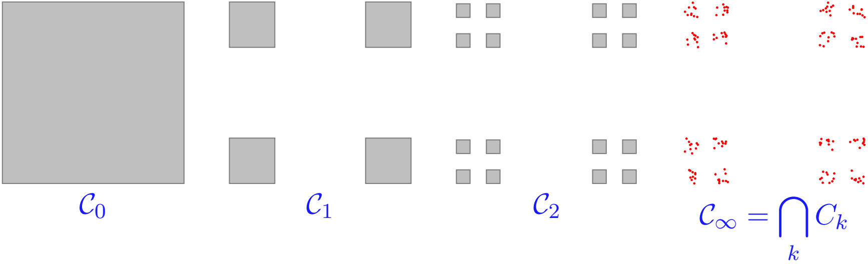

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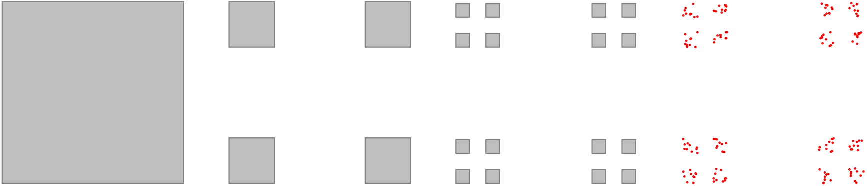

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Besicovitch (1929-1939)

Suppose $E \subset \mathbb{R}^2$ and $0 < \mathcal{H}^1(E) < \infty$. Then

$$E = R \cup P;$$

- R is 1-rectifiable.
- P is 1−purely unrectifiable.

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- R is n-rectifiable.
- P is $n-$purely unrectifiable.

Characterizing rectifiable sets

One of the main objectives of geometric measure theory consists in characterizing rectifiable sets in terms of

A The existence of approximate tangent n-planes.

B The existence of densities.

C The size of projections.

⋆ A + B + C are due to Besicovitch, Federer, Mattila, Preiss, ...

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By using the FORCE of PDE theory?

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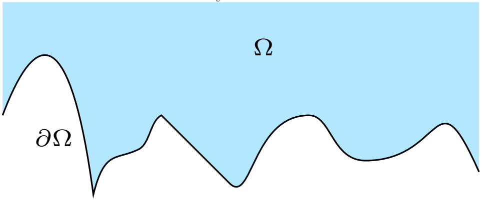
Harmonic measure

- $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, connected and open.

- Harmonic measure $\{\omega^X\}_{X \in \Omega}$ family of probabilities on $\partial\Omega$ called harmonic measure of Ω with a pole at $X \in \Omega$ such that

$$u(X) = \int_{\partial\Omega} f(x)\, d\omega^X(x) \quad \text{solves} \quad (D) \begin{cases} \mathcal{L}u = 0 \text{ in } \Omega \\ u\big|_{\partial\Omega} = f \in C_c(\partial\Omega). \end{cases}$$

Courtesy of Chema Martell



- Surface ball

$$\Delta(x,r) = B(x,r) \cap \partial\Omega,\ x \in \partial\Omega.$$

- $\sigma = \mathcal{H}^n\big|_{\partial\Omega}$.

- $\partial\Omega$ is $n-$Ahlfors-David regular (ADR) if

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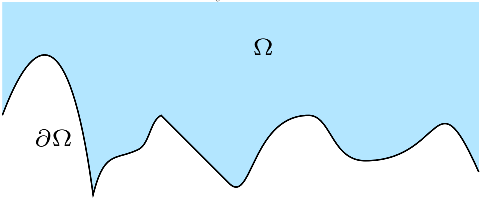

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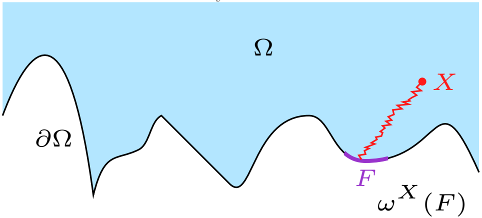

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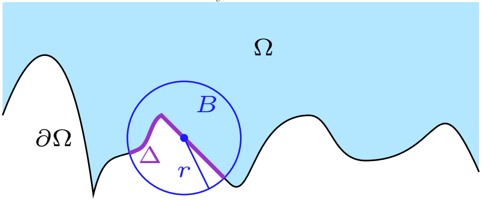

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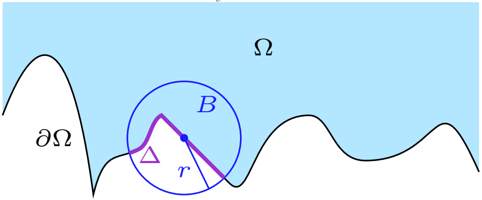

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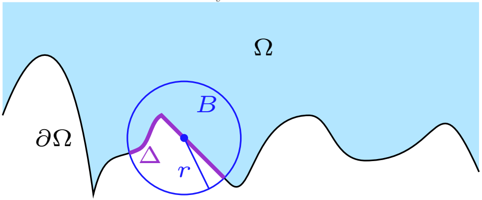

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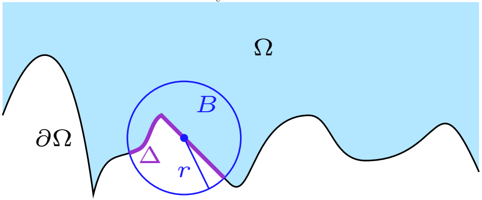

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Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain and let ω be the harmonic measure for Ω.

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Under what conditions, we have

$$\text{(A) } \omega \ll \mathcal{H}^n|_{\partial\Omega}? \qquad \text{and/or} \qquad \text{(B) } \mathcal{H}^n|_{\partial\Omega} \ll \omega?$$

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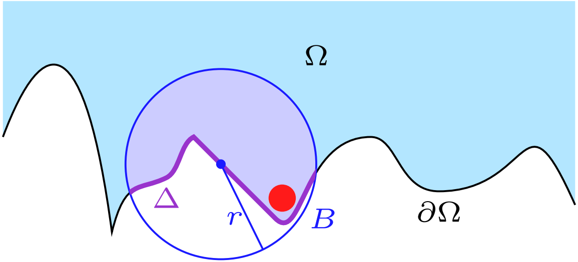
- Openness $\rightsquigarrow$ Corkscrew condition (CS).
- Path-connectedness $\rightsquigarrow$ Harnack chain condition (HC).



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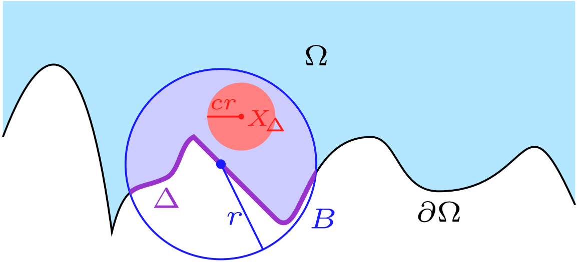

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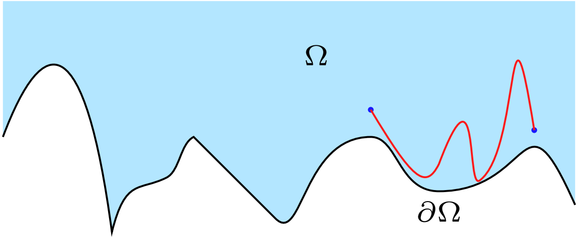

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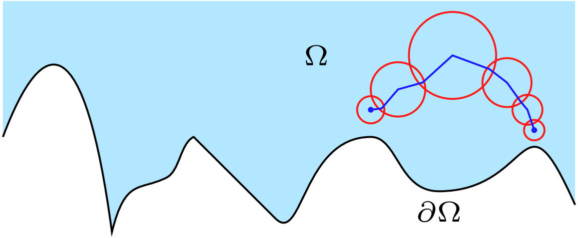

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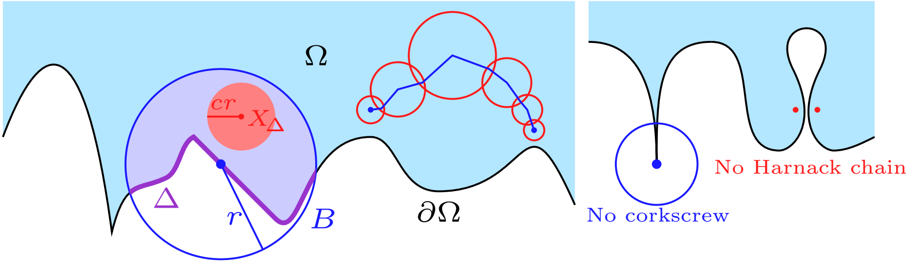

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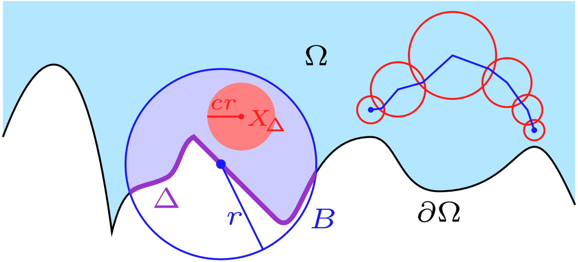

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- Ω is NTA $\equiv$ $\left\{ \begin{array}{l} \bullet\ \textbf{Interior}\ \text{Corkscrew and Harnack Chain.} \\ \bullet\ \textbf{Exterior}\ \text{Corkscrew.} \end{array} \right.$

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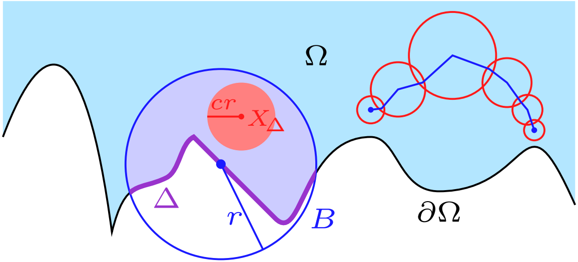

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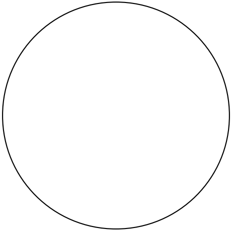
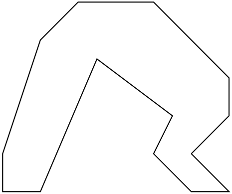
- Ω is NTA $\equiv$
  - **Interior** Corkscrew and Harnack Chain.
  - **Exterior** Corkscrew.
- Ω is 1-sided NTA $\equiv$ **Interior** Corkscrew and Harnack Chain.

Examples of such domains

Smooth Domains



Lipschitz Domains



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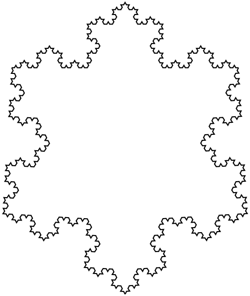

- NTA domains need not be graph domains or of finite perimeter.

A_∞ and A_∞^{weak} conditions

Let $E \subset \mathbb{R}^{n+1}$ be ADR set and let $\Delta_0 = E \cap B(z,r)$, $z \in E$.

A_∞ Condition

$\omega \in A_\infty(\Delta_0)$ with respect to $\mathcal{H}^n$ if there exist C and θ such that for all $\Delta = B(x,r') \cap E$ where $x, \in E$ and $B(x,r') \subset B(z,r)$ one has

$$\frac{\omega(F)}{\omega(\Delta)} \leq C \left(\frac{\mathcal{H}^n(F)}{\mathcal{H}^n(\Delta)} \right)^\theta \text{ for every Borel set } F \subset \Delta.$$

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Global results in higher dimension

- **Dahlberg**(1977): Ω is a Lipschitz domain then $\omega \in A_\infty(\mathcal{H}^n|_{\partial\Omega})$.
- **David and Jerison**(1990);
- **Semmes**(1989):

If Ω is NTA and $\partial\Omega$ is ADR then $\omega \in A_\infty(\mathcal{H}^n|_{\partial\Omega})$.

- **Badger**(2012): If $\Omega \subset \mathbb{R}^{n+1}$ is NTA with $\mathcal{H}^n(\partial\Omega) < \infty$ then

$$\mathcal{H}^n|_{\partial\Omega} \ll \omega \qquad \text{and} \qquad \omega \ll \mathcal{H}^n|_A$$

where

$$A = \left\{ x \in \partial\Omega;\ \liminf_{r\to 0} \frac{\mathcal{H}^n(\partial\Omega \cap B(x,r))}{r^n} < \infty \right\}.$$

⋆ Portions of the boundary should be contained in a nice set(like a graph or curve).

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Global results in higher dimension

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Theorem

Let Ω be 1-sided NTA and $\partial\Omega$ ADR. TFAE

1. $\partial\Omega$ is Uniformly Rectifiable[=ADR+Big Pieces of Lipschitz Images]
2. Ω is NTA domain (and therefore it is chord-arc domain).
3. $w \in A_\infty(\mathcal{H}^n|_{\partial\Omega})$.
4. $w \in A_\infty^{\text{weak}}(\mathcal{H}^n|_{\partial\Omega})$.

2 $\Longrightarrow$ 3 by David and Jerison and independently by Semmes.

4 $\Longrightarrow$ 1 by Hofmann, Martell, and Uriarte-Tuero.

3 $\Longrightarrow$ 4 is trivial.

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Theorem A (A., Badger, Hofmann, Martell)

Let Ω be 1-sided NTA and $\partial\Omega$ be ADR. TFAE;

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3. *$\sigma \ll \omega$ on $\partial\Omega$.*
4. *$\partial\Omega \overset{a.e.}{=} \bigcup_N F_N$ where $F_N = \partial\Omega_N \cap \partial\Omega$, $\Omega_N \subset \Omega$ is chord-arc.*
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4 $\Longrightarrow$ 1 as the boundary of any chord arc domain is rectifiable.

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1 $\Longrightarrow$ 2 by existence of approximate tangent planes.

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2 $\Longrightarrow$ 4 by constructing certain sawtooth domains which are bounded chord-arc subdomains of Ω.

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4 $\Longrightarrow$ 3 straightforward use of the maximum principle.

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3 $\Longrightarrow$ 4 by showing that some family of bad cubes (for which the exterior corkscrew condition fails) satisfies a Carleson packing condition. From there, we obtain that another suitable family of sawtooth domains are chord-arc domains.

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2 $\Longrightarrow$ 5 by using a variant of the Dahlberg-Jerison-Kenig sawtooth lemma and a certain projection operator.

Elliptic Operators

- $Lu(X) = \operatorname{div}(A\nabla u)(X)$, $X \in \Omega$.

- $A(X) = (a_{ij}(X))$ Real, Bounded, Symmetric, Uniformly Elliptic;

$$A(X)\xi\cdot\xi \geq \Lambda^{-1}|\xi|^2 \text{ and } |A(X)\xi\cdot\eta| \leq \Lambda|\xi||\eta|.$$

- $A \in \operatorname{Lip}_{\operatorname{loc}}(\Omega)$.

- ∇A satisfies a natural qualitative Carleson condition;

$$\sup_{\Delta\subset\partial\Omega} \frac{1}{\sigma(\Delta)} \iint_{\Delta} \left(\sup_{Z\in B(X,\delta(X)/2)} |\nabla A(Z)| \right) dx < \infty.$$

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Let Ω be 1-sided NTA and $\partial\Omega$ ADR. TFAE

1. $\partial\Omega$ *is Rectifiable.*
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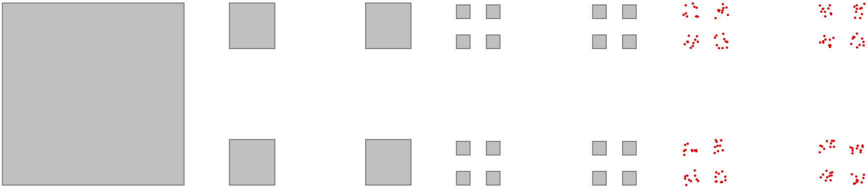
Let $\mathcal{C}_\infty$ be the "four-corner Cantor Set".



$\mathcal{C}_0$ $\mathcal{C}_1$ $\mathcal{C}_2$ $\mathcal{C}_\infty$

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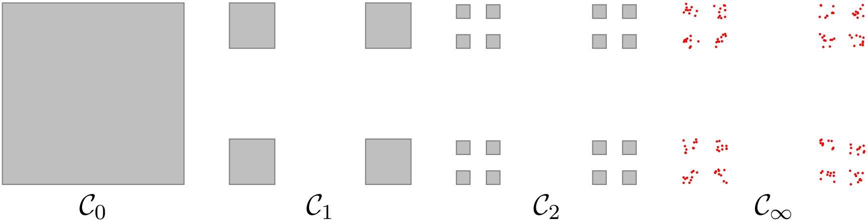

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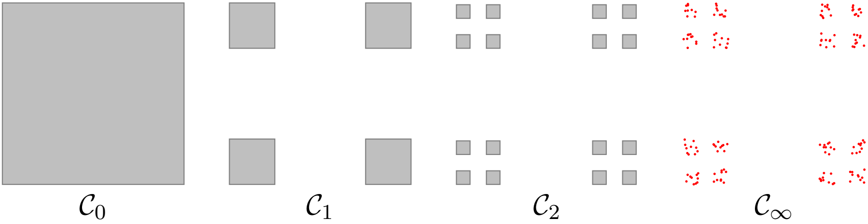

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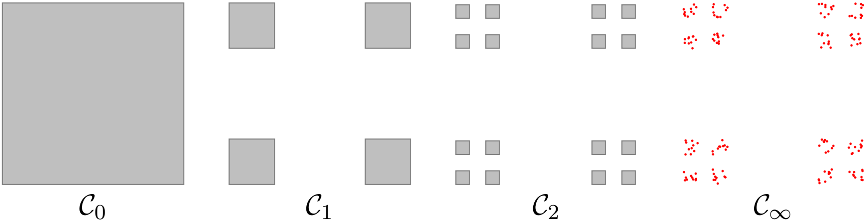

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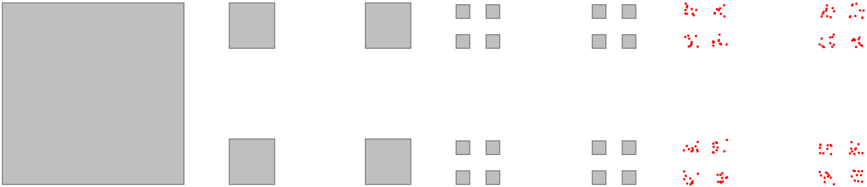

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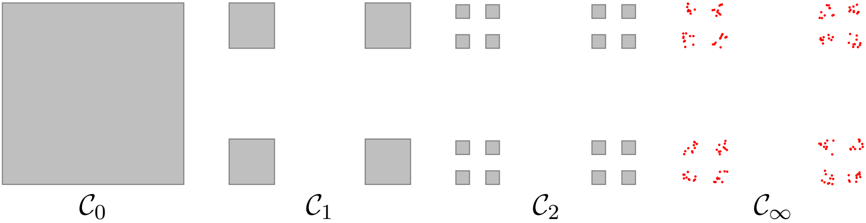

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All results requires some strong connectivity hypothesis;

1. Simply Connected or 2. Harnack Chain or 3. Corkscrew

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- $E \subset \mathbb{R}^{n+1}$, $n \geq 1$, closed set with locally finite $\mathcal{H}^n$-measure.

Let E_* be the realtively open set

$$E_* = \left\{ x \in E : \inf_{\substack{y \in B(x,\rho)\cap E \\ 0<r<\rho}} \frac{\mathcal{H}^n(B(y,r)\cap E)}{r^n} > 0, \text{ for some } \rho > 0 \right\}.$$

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Weakening the interior corkscrew condition

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Interior Measure Theoretic Boundary

The Interior Measure Theoretic Boundary $\partial_+\Omega$ is defined as

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Let Ω_α be the domain above the graph of the function $|\cdot|^\alpha$, $\alpha \in (0,\infty) \setminus \{1\}$;

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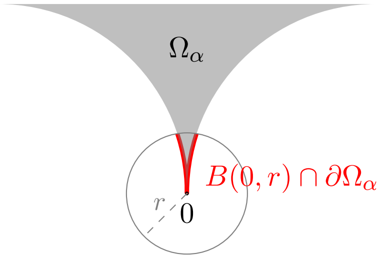

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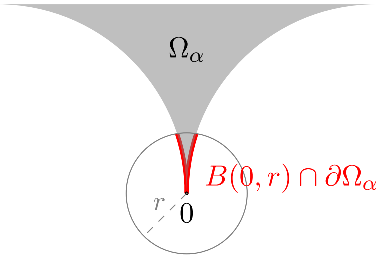

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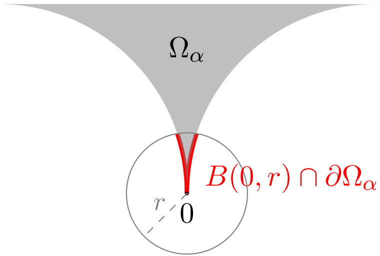

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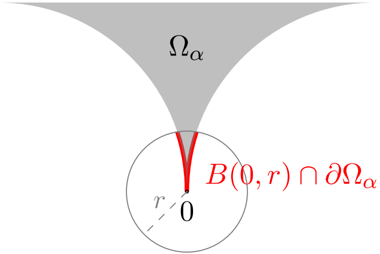

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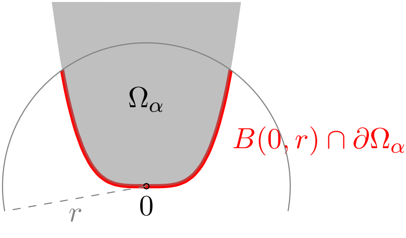

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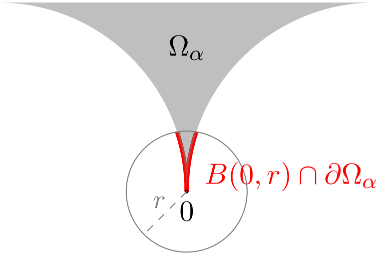

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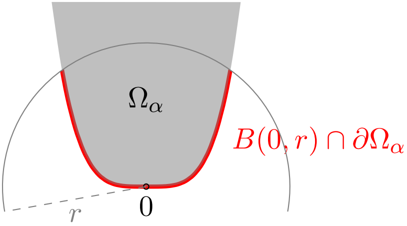

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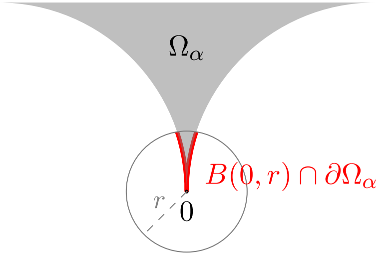

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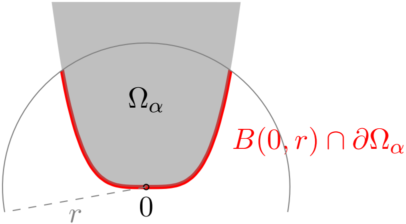

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A Theorem of McMillan in the plane

The Truncated cone $\Gamma_{h,\alpha}(z)$ is defined as

$$\Gamma_{h,\alpha}(z) := \{x : \ |z-x| < \alpha(1-|x|) < \alpha h\}.$$

A point $z \in \partial\Omega$ is called a Cone point if there is a truncated open cone $\Gamma_{h,\alpha}(z)$ with vertex at z such that $\Gamma_{h,\alpha}(z) \subset \Omega$.

- $K = K(\Omega) = \{\text{Cone points for } \Omega\}$.

Theorem[McMillan]

Let Ω be a bounded simply connected domain in the plane. Then

- K is a Borel set with $\sigma-$finite $\mathcal{H}^1$ measure,
- For $E \subset K$,

$$\omega(E) = 0 \Longleftrightarrow \mathcal{H}^1(E) = 0.$$

A. At almost every $w \in K$, $\partial\Omega$ has an inner tangent.

B. One can construct $\Omega' \subset \Omega$ with a rectifiable boundary such that

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Covering of E with boundaries of bounded Lipschitz domains

Theorem D (A., Bortz, Hofmann, Martell)

- *Let $E \subset \mathbb{R}^{n+1}$, $n \geq 1$, be a closed set,*
- *E have locally finite $\mathcal{H}^n$-measure,*
- *E satisfy the WLADR condition.*

Then

$$E \text{ is } n\text{-rectifiable} \iff E \subset Z \cup \Big(\bigcup_j \partial\Omega_j \Big).$$

- (A) *$\{\Omega_j\}_j$ is a countable collection of bounded Lipschitz domains,*
- (B) *$\Omega_j \subset \mathbb{R}^{n+1} \setminus E$ for every j,*
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Rectifiability implies absolute continuity

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Absolute continuity should be understood in the following sense;

$$\mathcal{H}^n|_E \ll \widetilde{\omega} := \sum_{k\geq 1} 2^{-k} \omega_k,$$

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- Let $\{\Omega_i\}$ be the bounded Lipschitz domains by Theorem ?? for E.
- Let $F \subset E$ be such that $\mathcal{H}^n(F) > 0$. Need to show $\omega(F) > 0$.
- Then there exists Ω_j such that $\mathcal{H}^n(F \cap \partial\Omega_j) > 0$.
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- Let ω^X be the harmonic measure for $\mathbb{R}^{n+1} \setminus E$ with pole at X.
- By the maximum principle and Dahlberg's result it follows that

$$\omega^X(F) \geq \omega_{\Omega_j}^X(F \cap \partial\Omega_j) > 0.$$

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$$\omega^X(F) \geq \omega^X_{\Omega_j}(F \cap \partial\Omega_j) > 0.$$

□

Proof of Theorem ?? assuming Theorem ??

- Let $\{\Omega_i\}$ be the bounded Lipschitz domains by Theorem ?? for E.
- Let $F \subset E$ be such that $\mathcal{H}^n(F) > 0$. Need to show $\omega(F) > 0$.
- Then there exists Ω_j such that $\mathcal{H}^n(F \cap \partial\Omega_j) > 0$.
- Pick $X \in \Omega_j \subset \mathbb{R}^{n+1} \setminus E$; $\omega^X_{\Omega_j}$ be the harmonic measure for Ω_j.
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Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be an open and connected set. Let

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Then

$$\partial\Omega \text{ is } n\text{-rectifiable} \iff \partial\Omega \subset Z \cup \left(\bigcup_j \partial\Omega_j^{int} \right),$$

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Local Results

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Higher dimensional version of McMillan's Theorem

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Then

$$\partial\Omega = R \cup P$$

where

1. R *is* $n-$*rectifiable such that* $\mathcal{H}^n|_R \ll \omega$.
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$$\mathcal{H}^n|_{\partial\Omega} = \mathcal{H}^n|_{\mathrm{ac}} + \mathcal{H}^n|_{\mathrm{s}} = \mathcal{H}^n|_F + \mathcal{H}^n|_{\partial\Omega\setminus F}$$

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- $\omega(\partial\Omega \setminus F) = 0$ (since $\mathcal{H}^n|_{\mathrm{s}} = \mathcal{H}^n|_{\partial\Omega\setminus F} \perp \omega$).
- As F is $n-$rectifiable by [AHM^{3}TV] (As $\mathcal{H}^n \ll \omega$ on $\partial\Omega$).
- Need to show $\partial\Omega \setminus F$ is purely $n-$unrectifiable.
- Suppose $\exists$ $n-$rectifiable Borel set F' with $\mathcal{H}^n(F' \cap (\partial\Omega \setminus F) > 0$.
- Apply Theorem **??** to rectifiable set $F' \cap (\partial\Omega \setminus F)$ to get $\omega(F' \cap (\partial\Omega \setminus F) > 0$ which contradicts with $\omega(\partial\Omega \setminus F) = 0$. ↯

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$$\Sigma_k = \left\{(x,t) \in \mathbb{R}^{n+1}_+ : \ t = 2^{-k}, |x| \geq 2^{-k}\right\}.$$

Let

$$\Omega := \mathbb{R}^{n+1}_+ \setminus \left(\bigcup_{k=1}^{\infty} \Sigma_k\right).$$

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Proof of covering of E with boundaries of bounded Lipschitz domains

Step 0: Rectifiability implies linear approximability:

Theorem[Mattila]

Let $E \subset \mathbb{R}^{n+1}$ be a n-rectifiable set such that $\mathcal{H}^n|_E$ is locally finite. Then there exists $E_0 \subset E$ with $\mathcal{H}^n(E_0) = 0$ such that if $x \in E \setminus E_0$ the following holds:

For every $\eta > 0$ there exist positive numbers $r_x = r_x(\eta)$ and $\lambda_x = \lambda_x(\eta)$ and a n-dimensional affine subspace $\mathcal{P}_x = \mathcal{P}_x(\eta)$ such that for all $0 < r < r_x$

1. $\mathcal{H}^n(E \cap B(y, \eta r)) \geq \lambda_x r^n, \quad \text{for } y \in \mathcal{P}_x \cap B(x, r)$
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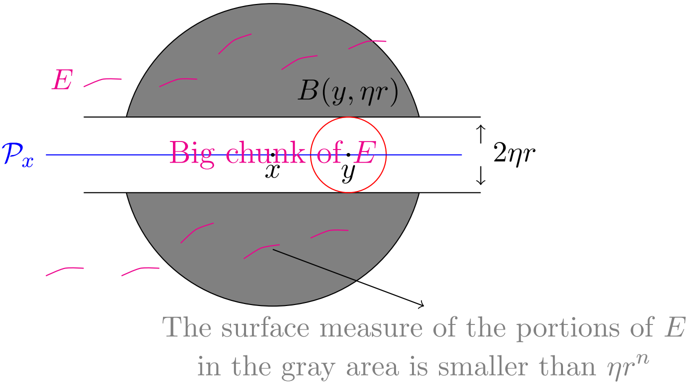

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① $\rightsquigarrow$ There is no big hole in E near $\mathcal{P}_x \cap B(x,r)$.

② $\mathcal{H}^n\big((E \cap B(x,r)) \setminus P_x^{(\eta r)}\big) < \eta r^n$.

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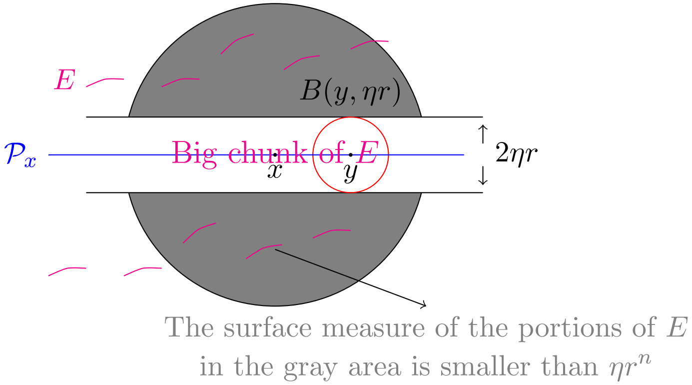

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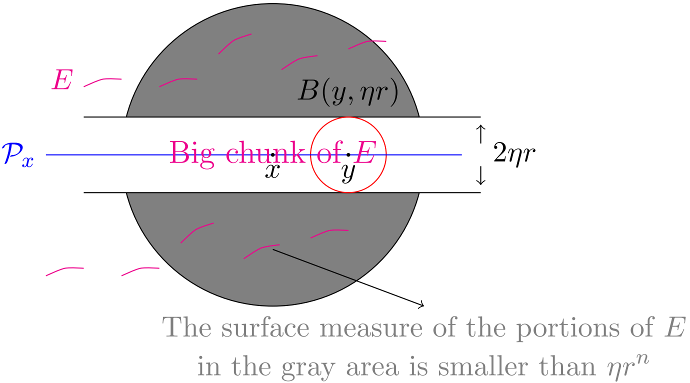

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Let $x \in E \setminus E_0$ with constants $\lambda_x(\eta)$ and $r_x(\eta)$. For every $0 < \eta < \eta_0(\lambda_x) := \min\{2^{-4\,n}, \lambda_x^2\}$;

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⋆ $\alpha(\eta) \to \pi$ as $\eta \to 0^+$.

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Let $x \in E \setminus E_0$ with constants $\lambda_x(\eta)$ and $r_x(\eta)$. For every $0 < \eta < \eta_0(\lambda_x) := \min\{2^{-4\,n}, \lambda_x^2\}$;

- there exists a two sided truncated cone $\Gamma_{h,\alpha}(x)$ with vertex at x with,

A height $h(\eta) := \eta^{\frac{1}{4n}} \min\{r_x(\eta), r_x\}$,

B aperture $\alpha(\eta) := 2 \arctan\left(\eta^{-\frac{1}{4n}}/2\right) > \pi/2$,

C DOES NOT meet with E.

⋆ $\alpha(\eta) \to \pi$ as $\eta \to 0^+$.

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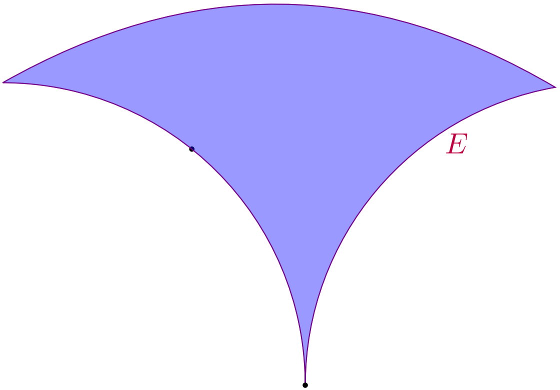

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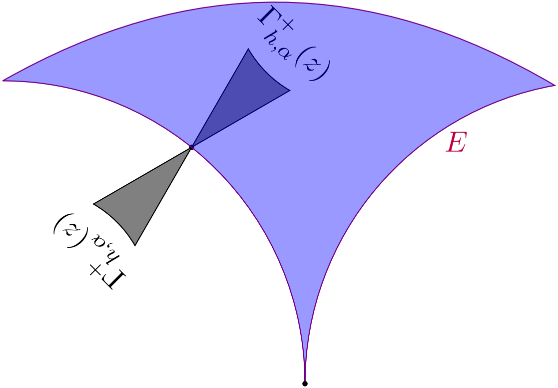

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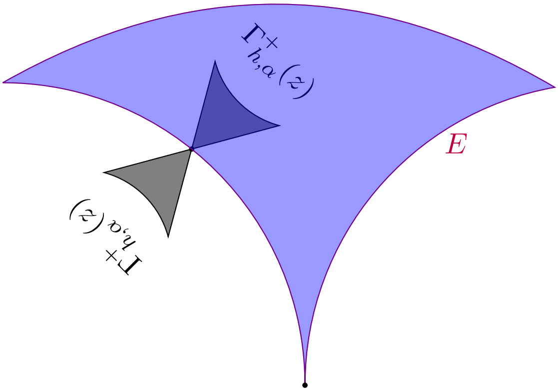

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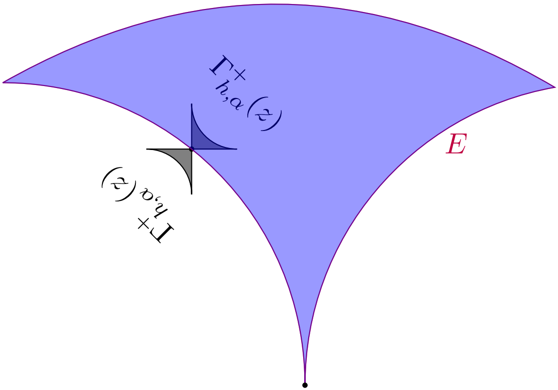

Step 2: One of the cone must be interior:

- Consider $\Omega = \mathbb{R}^{n+1} \setminus E$.

- Let $x \in \partial\Omega \setminus E_0 = E \setminus E_0$ with constants $\lambda_x(\eta)$ and $r_x(\eta)$ and for every $0 < \eta < \eta_0(\lambda_x) := \min\{2^{-4\,n}, \lambda_x^2\}$ there exist two sided truncated cone, $\Gamma^+_{h,\alpha}(x), \Gamma^-_{h,\alpha}(x)$

- Given $\epsilon > 0$ there exists $\tilde{\eta}_0 = \tilde{\eta}_0(\epsilon) < \eta_0(c_x)$ such that if $0 < \eta < \tilde{\eta}_0$ and

$$\limsup_{r \to 0^+} \frac{|B(x,r) \cap \Omega|}{|B(x,r)|} > \epsilon.$$

- Then one of the cones constructed must be in the interior of Ω.

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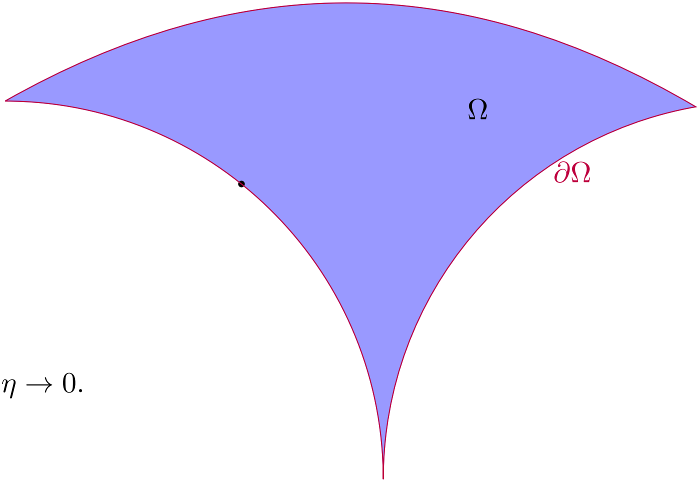

- A $\alpha \to \pi$ when $\eta \to 0$.

- C $\frac{\text{Red part}}{r^{n+1}} = \frac{|B(z,r) \setminus (\Gamma^+_{h,\alpha}(z) \cup \Gamma^-_{h,\alpha}(z))|}{|B(z,r)|} \to 0.$

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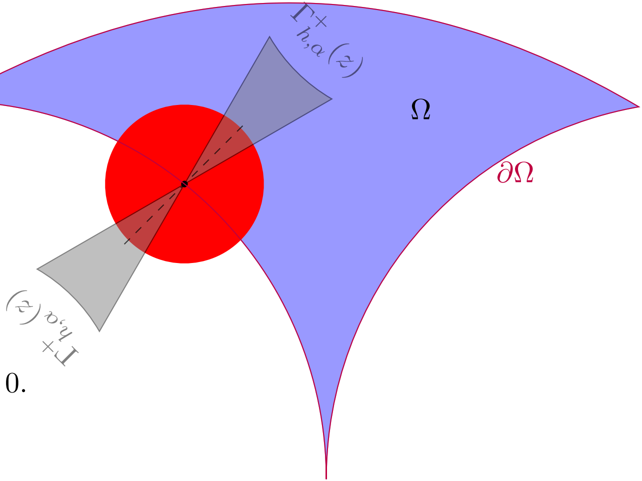

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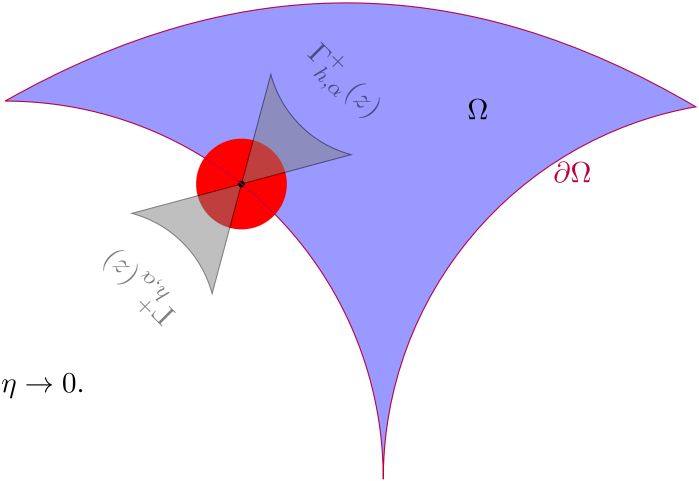

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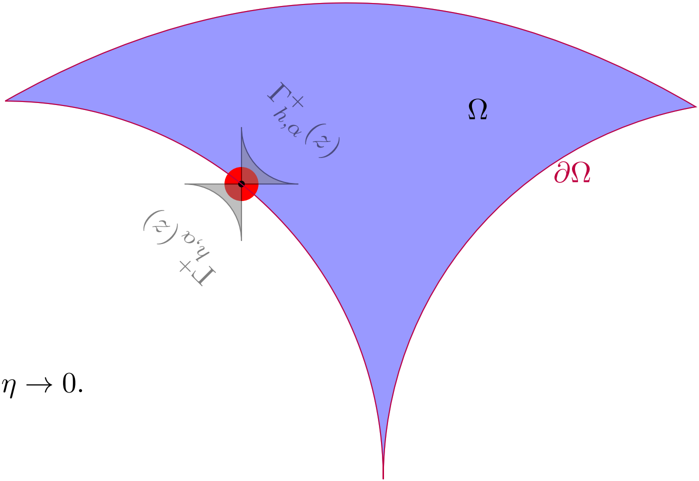

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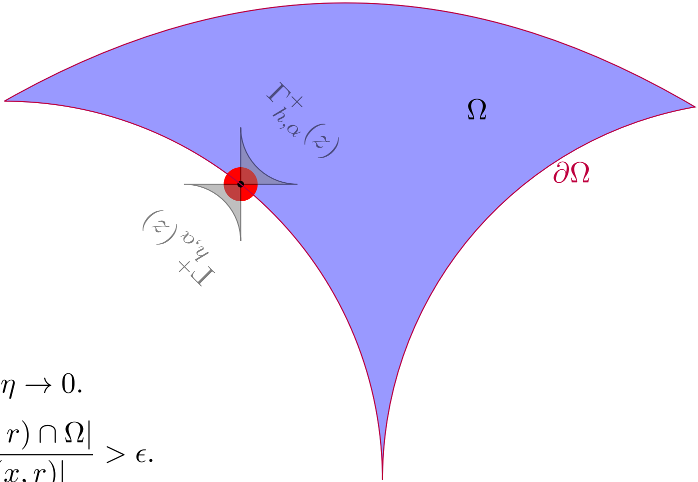

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Proof of Theorem ??

- Choose $\{\nu_m\}_{m=1}^{M} \subset \mathbb{S}^n$ (the unit sphere in $\mathbb{R}^{n+1}$) such that for every $\nu \in \mathbb{S}^n$ there exists ν_m, $1 \le m \le M$, such that angle$(\nu, \nu_m) < \pi/8$.
- Set $P_m := \nu_m^{\perp}$, $1 \le m \le M$.
- $E_* := \{x \in E;\ \mathcal{H}^n(B(y,r) \cap E) \ge c_x\, r^n,\ \forall\, y \in B(x, \rho_x) \cap E,\ 0 < r \le \rho_x.\}$.
- For every $k \in N$ an $1 \le m \le M$ we set

$$G(k,m) := \Big\{x \in E_* \backslash E_0 : \max\{c_x, \rho_x, r_x\} > 2^{-k},\ \text{angle}(P_m, P_x) < \pi/8\Big\}.$$

- Notice that setting $Z = (E \setminus E_*) \cup E_0$ we have that $\mathcal{H}^n(Z) = 0$. Also,

$$E = Z \cup \Big(\bigcup_{m=1}^{M} \bigcup_{k \in N} G(k,m) \Big).$$

- Need to show: $G(k,m)$ can be covered by a countable union of boundaries of bounded Lipschitz domains missing E. $\square$

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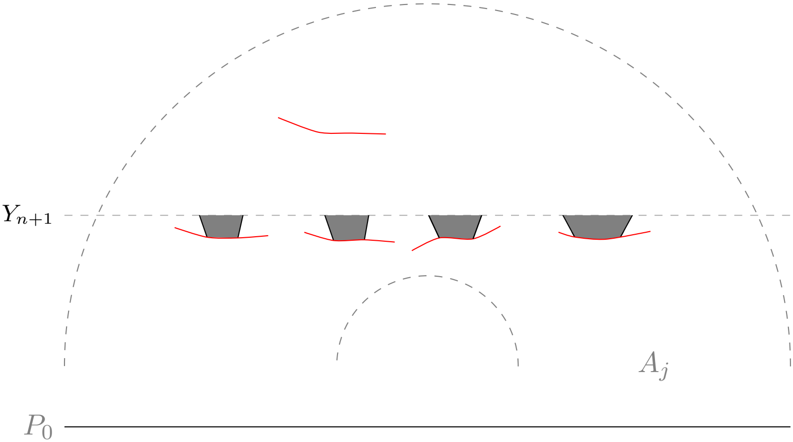

Y_{n+1}
A_j
P_0

THANKS!