# Perturbations of elliptic operators on rough domains


Murat Akman

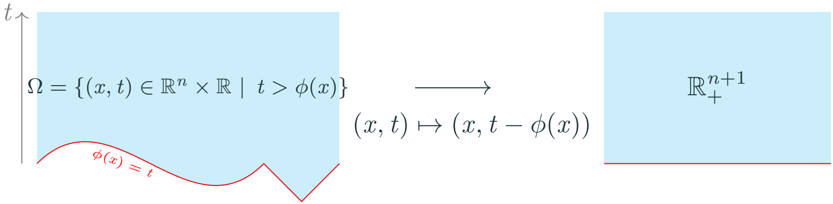
Joint work with Steve Hofmann, José María Martell, and Tatiana Toro



Trimester Seminar Series: Interactions between GMT, Singular integrals, and PDEs
The Hausdorff Research Institute for Mathematics
18 March 2022

# Motivation

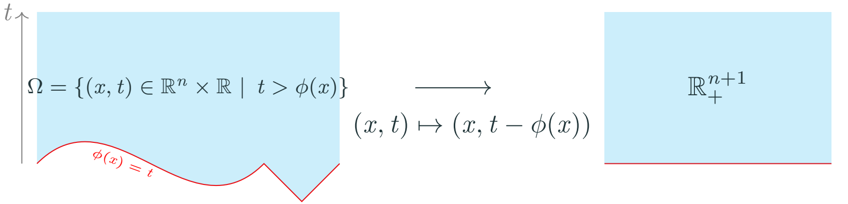
Let $\Omega$ be the domain above the graph of a Lipschitz function $\phi$.




$$-\Delta u = -\operatorname{div}(I \nabla u) = 0 \dashrightarrow Lu = -\operatorname{div}(A \nabla u) = 0$$

# Motivation

Let $\Omega$ be the domain above the graph of a Lipschitz function $\phi$.

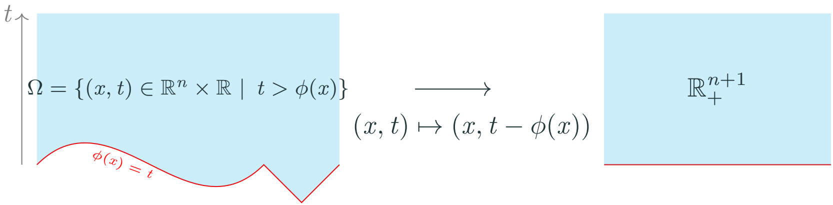



$$-\Delta u = -\operatorname{div}(I\nabla u) = 0 \dashrightarrow Lu = -\operatorname{div}(A\nabla u) = 0$$

- $A$ depends on the Jacobian of the change of variables, hence are **bounded** and **measurable**, but NOT any more regular.

# Motivation

Let $\Omega$ be the domain above the graph of a Lipschitz function $\phi$.




$$-\Delta u = -\operatorname{div}(I \nabla u) = 0 \dashrightarrow Lu = -\operatorname{div}(A \nabla u) = 0$$

- $A$ depends on the Jacobian of the change of variables, hence are **bounded** and **measurable**, but NOT any more regular.
- $A$ is **uniformly elliptic** matrix; there exists constant $\Lambda \geq 1$ such that

$$\Lambda^{-1}|\xi|^2 \leq A(X)\xi \cdot \xi, \qquad |A(X)\xi \cdot \eta| \leq \Lambda |\xi|\,|\eta|$$

for all $\xi, \eta \in \mathbb{R}^{n+1}$ and for almost every $X \in \mathbb{R}^{n+1}_+$.

# Elliptic operators

- Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set.
- Let $L$ be a second order divergence form real elliptic operator defined in $\Omega$

$$Lu = -\operatorname{div}(A\nabla u)$$

Here the coefficient matrix $A = A(X)$ is $A = (a_{i,j}(\cdot))_{i,j=1}^{n+1}$ is real, symmetric, with $a_{i,j} \in L^\infty(\Omega)$ and is uniformly elliptic, that is, there exists a constant $\Lambda \geq 1$ such that

$$\Lambda^{-1}|\xi|^2 \leq A(X)\xi \cdot \xi, \qquad |A(X)\xi \cdot \eta| \leq \Lambda|\xi|\,|\eta|$$

for all $\xi, \eta \in \mathbb{R}^{n+1}$ and for almost every $X \in \Omega$.

# Elliptic operators

- Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set.
- Let $L$ be a second order divergence form real elliptic operator defined in $\Omega$

$$Lu = -\operatorname{div}(A\nabla u)$$

Here the coefficient matrix $A = A(X)$ is $A = (a_{i,j}(\cdot))_{i,j=1}^{n+1}$ is real, symmetric, with $a_{i,j} \in L^\infty(\Omega)$ and is uniformly elliptic, that is, there exists a constant $\Lambda \geq 1$ such that

$$\Lambda^{-1}|\xi|^2 \leq A(X)\xi \cdot \xi, \qquad |A(X)\xi \cdot \eta| \leq \Lambda|\xi|\,|\eta|$$

for all $\xi, \eta \in \mathbb{R}^{n+1}$ and for almost every $X \in \Omega$.

- $Lu = 0$ in $\Omega$ if $u \in W^{1,2}_{\text{loc}}(\Omega)$ and

$$\int \langle A\nabla u, \nabla\psi\rangle \, dX = 0 \quad \text{whenever} \quad \psi \in C_0^\infty(\Omega).$$

# Elliptic measure

- $\Omega$ is called regular for the operator $L$ if for every $f \in C_c(\partial\Omega)$, there exists a (generalized) solution $u = u_f \in C(\bar{\Omega})$ which solves

$$\begin{cases} Lu = -\operatorname{div}(A\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

# Elliptic measure

- $\Omega$ is called regular for the operator $L$ if for every $f \in C_c(\partial\Omega)$, there exists a (generalized) solution $u = u_f \in C(\bar{\Omega})$ which solves

$$\begin{cases} Lu = -\operatorname{div}(A\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

**Representation formula**

Elliptic measure $\{\omega_L^X\}_{X\in\Omega}$ is the unique probability measure s.t.

$$u(X) = \int_{\partial\Omega} f(z)\, d\omega_L^X(z).$$

# Elliptic measure

- $\Omega$ is called **regular** for the operator $L$ if for every $f \in C_c(\partial\Omega)$, there exists a (generalized) solution $u = u_f \in C(\bar{\Omega})$ which solves

$$\begin{cases} Lu = -\operatorname{div}(A\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

### Representation formula

Elliptic measure $\{\omega_L^X\}_{X\in\Omega}$ is the unique probability measure s.t.

$$u(X) = \int_{\partial\Omega} f(z)\, d\omega_L^X(z).$$

- When $\Omega$ is the unit ball in $\mathbb{R}^{n+1}$ and $A = I$ (i.e., Laplace operator)

$$\omega^X(E) = \int_E \frac{1-|X|^2}{|X-Y|^{n+1}} \frac{d\sigma(Y)}{\sigma(\mathbb{S}^n)} \quad \text{whenever } E \subset \mathbb{S}^n.$$

# Elliptic measure

▶ $\Omega$ is called regular for the operator $L$ if for every $f \in C_c(\partial\Omega)$, there exists a (generalized) solution $u = u_f \in C(\bar{\Omega})$ which solves

$$\begin{cases} Lu = -\operatorname{div}(A\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

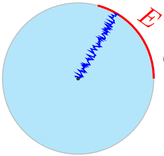
**Representation formula**

Elliptic measure $\{\omega_L^X\}_{X\in\Omega}$ is the unique probability measure s.t.

$$u(X) = \int_{\partial\Omega} f(z)\, d\omega_L^X(z).$$

▶ When $\Omega$ is the unit ball in $\mathbb{R}^{n+1}$ and $A = I$ (i.e., Laplace operator)

$$\omega^X(E) = \int_E \frac{1-|X|^2}{|X-Y|^{n+1}} \frac{d\sigma(Y)}{\sigma(\mathbb{S}^n)} \quad \text{whenever } E \subset \mathbb{S}^n.$$




$$\omega^0(E) = \int_E \frac{1-|0|^2}{|0-Y|^2} \frac{d\sigma(Y)}{\sigma(\mathbb{S}^1)} = \frac{\sigma(E)}{2\pi} = \frac{\text{arclength}(E)}{2\pi}.$$

## A question of Dahlberg

- Let $A_0$ and $A$ be real and uniformly elliptic.

Consider

"**Good**" operator
$$\begin{cases} L_0 u = -\operatorname{div}(A_0 \nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

think of $A_0 = I \to$ we have Laplacian.

"**Perturbed**" operator
$$\begin{cases} L u = -\operatorname{div}(A \nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

# A question of Dahlberg

- Let $A_0$ and $A$ be real and uniformly elliptic.

Consider

"**Good**" operator
$$\begin{cases} L_0 u = -\operatorname{div}(A_0 \nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

think of $A_0 = I \to$ we have Laplacian.

"**Perturbed**" operator
$$\begin{cases} Lu = -\operatorname{div}(A \nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

**Question (Dahlberg '84)**
Suppose that we have "good estimates" for the Dirichlet problem for $L_0$, **under what optimal conditions**, are those good estimates transferred to the Dirichlet problem for $L$?

# A question of Dahlberg

- Let $A_0$ and $A$ be real and uniformly elliptic.

Consider

"**Good**" operator
$$\begin{cases} L_0 u = -\operatorname{div}(A_0 \nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

think of $A_0 = I \to$ we have Laplacian.

"**Perturbed**" operator
$$\begin{cases} Lu = -\operatorname{div}(A \nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

**Question (Dahlberg '84)**
Suppose that we have "good estimates" for the Dirichlet problem for $L_0$, **under what optimal conditions**, are those good estimates transferred to the Dirichlet problem for $L$?

Good estimates: $\omega_{L_0} \in A_\infty(\sigma)$, $\omega_{L_0} \in \mathrm{RH}_p(\sigma)$, etc...

$\omega_{L_0}$ is the elliptic measure of $\Omega$ associated to the operator $L_0$.

$\omega_L$ is the elliptic measure of $\Omega$ associated to the operator $L$

# Dark side of the linear elliptic operators

- [Littman-Stampacchia-Weinberger '63] $\Omega$ is regular for **Laplacian** (i.e., $A_0 = I$) iff it is regular for any $L$.

# Dark side of the linear elliptic operators

- [Littman-Stampacchia-Weinberger '63] $\Omega$ is regular for **Laplacian** (i.e., $A_0 = I$) iff it is regular for any $L$.
- Fact: If $\Omega = \mathbb{R}^{1+1}_+$ and $A_0 = I$ so that we have Laplacian, then harmonic measure $\omega$ is *mutually* absolutely continuous with respect to surface measure $\sigma$ (i.e., $\sigma \ll \omega \ll \sigma$).

# Dark side of the linear elliptic operators

- [Littman-Stampacchia-Weinberger '63] $\Omega$ is regular for **Laplacian** (i.e., $A_0 = I$) iff it is regular for any $L$.
- Fact: If $\Omega = \mathbb{R}^{1+1}_{+}$ and $A_0 = I$ so that we have Laplacian, then harmonic measure $\omega$ is *mutually* absolutely continuous with respect to surface measure $\sigma$ (i.e., $\sigma \ll \omega \ll \sigma$).
- [Caffarelli-Fabes-Kenig '81] There exists an elliptic operator $L$, the pullback of the Laplacian via a quasiconformal mapping of the upper plane to itself, but $\omega_L \not\ll \sigma$.

# Dark side of the linear elliptic operators

- [Littman-Stampacchia-Weinberger '63] $\Omega$ is regular for **Laplacian** (i.e., $A_0 = I$) iff it is regular for any $L$.
- Fact: If $\Omega = \mathbb{R}^{1+1}_+$ and $A_0 = I$ so that we have Laplacian, then harmonic measure $\omega$ is *mutually* absolutely continuous with respect to surface measure $\sigma$ (i.e., $\sigma \ll \omega \ll \sigma$).
- [Caffarelli-Fabes-Kenig '81] There exists an elliptic operator $L$, the pullback of the Laplacian via a quasiconformal mapping of the upper plane to itself, but $\omega_L \not\ll \sigma$.
- [Modica-Mortola '81] There exists elliptic operator $L$

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y}\left(\alpha \frac{\partial}{\partial y}\right) \quad \text{where} \quad \alpha \in C(\bar{\Omega}) \cap C^\infty(\Omega)$$

where $\partial\Omega \in C^\infty$ in $\mathbb{R}^2$ such that $\omega_L \not\ll \sigma$.

# Dark side of the linear elliptic operators

- [Littman-Stampacchia-Weinberger '63] $\Omega$ is regular for **Laplacian** (i.e., $A_0 = I$) iff it is regular for any $L$.
- Fact: If $\Omega = \mathbb{R}^{1+1}_+$ and $A_0 = I$ so that we have Laplacian, then harmonic measure $\omega$ is *mutually* absolutely continuous with respect to surface measure $\sigma$ (i.e., $\sigma \ll \omega \ll \sigma$).
- [Caffarelli-Fabes-Kenig '81] There exists an elliptic operator $L$, the pullback of the Laplacian via a quasiconformal mapping of the upper plane to itself, but $\omega_L \not\ll \sigma$.
- [Modica-Mortola '81] There exists elliptic operator $L$

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y}\left(\alpha \frac{\partial}{\partial y}\right) \quad \text{where} \quad \alpha \in C(\bar{\Omega}) \cap C^\infty(\Omega)$$

where $\partial\Omega \in C^\infty$ in $\mathbb{R}^2$ such that $\omega_L \not\ll \sigma$.

- *It is not the domain $\Omega$ but the disagreement of $A$ and $A_0$ that we should have conditions on...*

# Reverse Hölder classes

If $\mu \ll \nu$ then $\exists g$, integrable, such that $d\mu = g d\nu$.

# Reverse Hölder classes

If $\mu \ll \nu$ then $\exists g$, integrable, such that $d\mu = g d\nu$.

- We say that $\mu \in \mathrm{RH}_q(\nu)$ for $1 < q < \infty$ if $\exists C > 0$ such that

$$\left( \frac{1}{\nu(\Delta)} \int_\Delta g^q d\nu \right)^{1/q} \leq C \frac{1}{\nu(\Delta)} \int_\Delta g d\nu$$

for every $\Delta = \Delta(x, r)$ centred on $\partial\Omega$ with $r \in (0, \mathrm{diam}(\partial\Omega))$.

## Reverse Hölder classes

If $\mu \ll \nu$ then $\exists g$, integrable, such that $d\mu = g d\nu$.

- We say that $\mu \in \mathsf{RH}_q(\nu)$ for $1 < q < \infty$ if $\exists C > 0$ such that

$$\left(\frac{1}{\nu(\Delta)}\int_\Delta g^q d\nu\right)^{1/q} \leq C\frac{1}{\nu(\Delta)}\int_\Delta g d\nu$$

for every $\Delta = \Delta(x, r)$ centred on $\partial\Omega$ with $r \in (0, \operatorname{diam}(\partial\Omega))$.

- The union of the $\mathsf{RH}_q$ classes is the $A_\infty$ class,

$$A_\infty(\nu) := \bigcup_{1<q<\infty} \mathsf{RH}_q(\nu).$$

# Reverse Hölder classes

If $\mu \ll \nu$ then $\exists g$, integrable, such that $d\mu = g d\nu$.

- We say that $\mu \in \mathrm{RH}_q(\nu)$ for $1 < q < \infty$ if $\exists C > 0$ such that

$$\left( \frac{1}{\nu(\Delta)} \int_\Delta g^q d\nu \right)^{1/q} \leq C \frac{1}{\nu(\Delta)} \int_\Delta g d\nu$$

for every $\Delta = \Delta(x, r)$ centred on $\partial\Omega$ with $r \in (0, \mathrm{diam}(\partial\Omega))$.

- The union of the $\mathrm{RH}_q$ classes is the $A_\infty$ class,

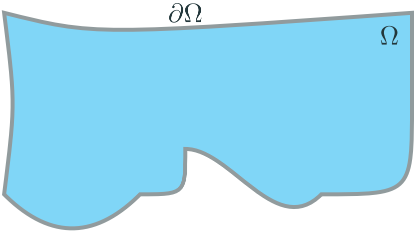
$$A_\infty(\nu) := \bigcup_{1<q<\infty} \mathrm{RH}_q(\nu).$$

- $\mu \in \mathrm{RH}_q(\nu)$, $1 < q < \infty$, is equivalent to the solvability of the $(D)_p$ Dirichlet problem

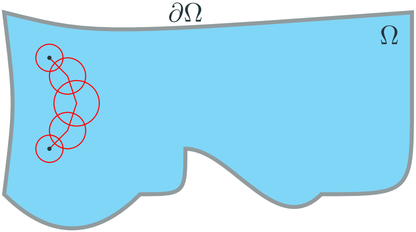
$$(D)_p \begin{cases} Lu = -\mathrm{div}(A\nabla u) = 0 & \text{in } \Omega, \\ u = f \in L^p(\nu) & \text{on } \partial\Omega, \\ \|N(u)\|_{L^p(\nu)} \leq C\|f\|_{L^p(\nu)} & \end{cases}$$

where $N(u)(X) = \sup\limits_{X \in \Gamma(X)} |u(Y)|$ and $1/p + 1/q = 1$.

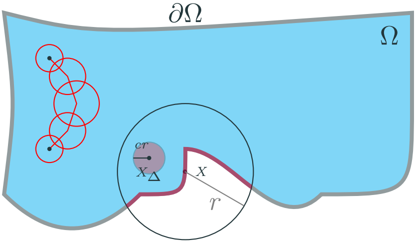
# Non-tangentially Accessible Domains(NTA)



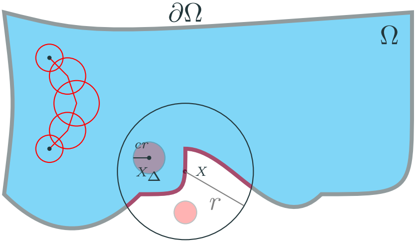
# Non-tangentially Accessible Domains(NTA)



# Non-tangentially Accessible Domains(NTA)

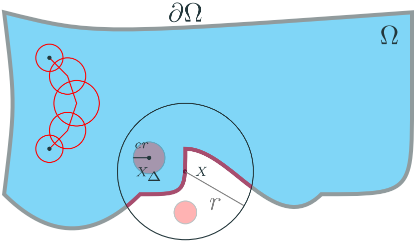


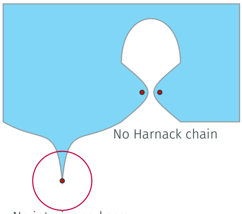
# Non-tangentially Accessible Domains(NTA)




- $\Omega$ is NTA $\equiv$ $\begin{cases} \text{▶ Interior Corkscrew and Harnack Chain.} \\ \text{▶ Exterior Corkscrew.} \end{cases}$

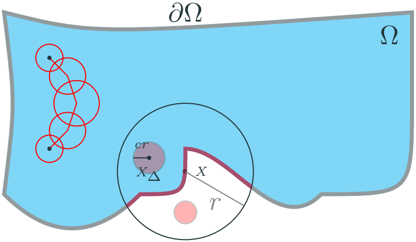
# Non-tangentially Accessible Domains(NTA)

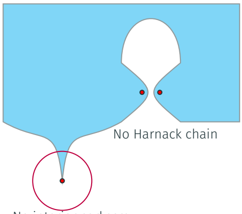



- $\Omega$ is NTA $\equiv \begin{cases} \blacktriangleright \text{ Interior Corkscrew and Harnack Chain.} \\ \blacktriangleright \text{ Exterior Corkscrew.} \end{cases}$

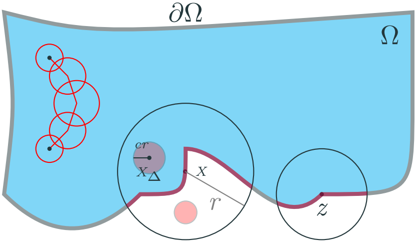
# Non-tangentially Accessible Domains(NTA)

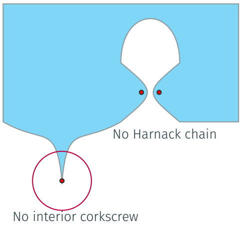



- $\Omega$ is NTA $\equiv$ $\begin{cases} \text{▶ Interior Corkscrew and Harnack Chain.} \\ \text{▶ Exterior Corkscrew.} \end{cases}$
- $\Omega$ is 1-sided NTA $\equiv$ **Interior** Corkscrew and Harnack Chain.

# Non-tangentially Accessible Domains(NTA)



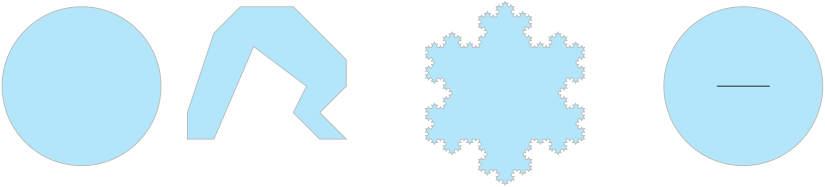




- $\Omega$ is NTA $\equiv$ $\left\{ \right.$
  - **Interior** Corkscrew and Harnack Chain.
  - **Exterior** Corkscrew.
- $\Omega$ is 1-sided NTA $\equiv$ **Interior** Corkscrew and Harnack Chain.
- $\partial\Omega$ is $n-$Ahlfors-David regular (ADR) if

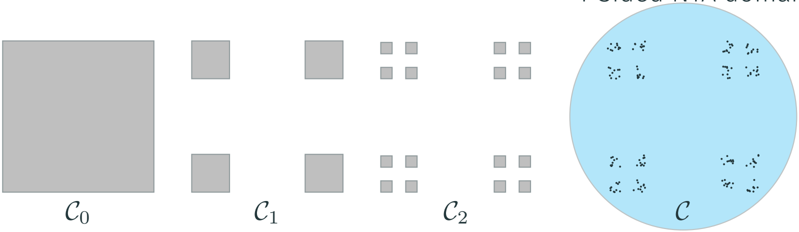
$$cr^n \le \sigma(\Delta(z,r)) \le cr^n \text{ whenever } z \in \partial\Omega \text{ and } r \in (0, \operatorname{diam}(\partial\Omega)).$$

# Examples of such domains

Smooth | Lipschitz | NTA | Semi-uniform



1-Sided NTA domain.




Smooth $\subsetneq$ Lipschitz $\subsetneq$ NTA $\subsetneq$ 1-sided NTA $\subsetneq$ Semi-uniform.

# Approach I

Let $\delta(X) = \mathrm{dist}(X, \partial\Omega)$, $\mathbf{a}(X) := \sup_{Y \in B(X,\delta(X)/2)} |A(Y) - A_0(Y)|$.

# Approach I

Let $\delta(X) = \mathrm{dist}(X, \partial\Omega)$, $\mathbf{a}(X) := \sup_{Y \in B(X, \delta(X)/2)} |A(Y) - A_0(Y)|$.

**Theorem (Dahlberg '77)**
Let $\Omega = B(0,1)$. Let $x \in \partial\Omega$ and $\Delta = \Delta(x,r) = B(x,r) \cap \partial\Omega$ and $T(\Delta) = B(x,r) \cap \Omega$. If $\frac{\mathbf{a}^2(X)}{\delta(X)}\, dX$ satisfies the vanishing Carleson measure condition, i.e.,

$$\lim_{r \to 0} \sup_{\Delta \subset \partial\Omega} \left\{ \frac{1}{\sigma(\Delta)} \int_{T(\Delta)} \frac{\mathbf{a}^2(X)}{\delta(X)}\, dX \right\}^{1/2} = 0$$

then $\omega_L \in \mathrm{RH}_p(\sigma)$ whenever $\omega_{L_0} \in \mathrm{RH}_p(\sigma)$.

# Approach I

Let $\delta(X) = \text{dist}(X, \partial\Omega)$, $\mathbf{a}(X) := \sup_{Y \in B(X, \delta(X)/2)} |A(Y) - A_0(Y)|$.

**Theorem (Dahlberg '77)**
Let $\Omega = B(0,1)$. Let $x \in \partial\Omega$ and $\Delta = \Delta(x,r) = B(x,r) \cap \partial\Omega$ and $T(\Delta) = B(x,r) \cap \Omega$. If $\frac{\mathbf{a}^2(X)}{\delta(X)}\, dX$ satisfies the vanishing Carleson measure condition, i.e.,

$$\lim_{r \to 0} \sup_{\Delta \subset \partial\Omega} \left\{ \frac{1}{\sigma(\Delta)} \int_{T(\Delta)} \frac{\mathbf{a}^2(X)}{\delta(X)}\, dX \right\}^{1/2} = 0$$

then $\omega_L \in \mathsf{RH}_p(\sigma)$ whenever $\omega_{L_0} \in \mathsf{RH}_p(\sigma)$.

- [R. Fefferman-Kenig-Pipher '91] Let $\Omega$ be a Lipschitz domain and if $\frac{\mathbf{a}^2(X)}{\delta(X)}\, dX$ satisfies a Carleson measure condition then $\omega_{L_0} \in A_\infty(\sigma)$ whenever $\omega_L \in A_\infty(\sigma)$.

# Approach I

Let $\delta(X) = \text{dist}(X, \partial\Omega)$, $\mathbf{a}(X) := \sup_{Y \in B(X,\delta(X)/2)} |A(Y) - A_0(Y)|$.

**Theorem (Dahlberg '77)**
Let $\Omega = B(0,1)$. Let $x \in \partial\Omega$ and $\Delta = \Delta(x,r) = B(x,r) \cap \partial\Omega$ and $T(\Delta) = B(x,r) \cap \Omega$. If $\frac{\mathbf{a}^2(X)}{\delta(X)}\,dX$ satisfies the vanishing Carleson measure condition, i.e.,

$$\lim_{r\to 0} \sup_{\Delta\subset\partial\Omega} \left\{ \frac{1}{\sigma(\Delta)} \int_{T(\Delta)} \frac{\mathbf{a}^2(X)}{\delta(X)}\,dX \right\}^{1/2} = 0$$

then $\omega_L \in \mathrm{RH}_p(\sigma)$ whenever $\omega_{L_0} \in \mathrm{RH}_p(\sigma)$.

- [R. Fefferman-Kenig-Pipher '91] Let $\Omega$ be a Lipschitz domain and if $\frac{\mathbf{a}^2(X)}{\delta(X)}\,dX$ satisfies a Carleson measure condition then $\omega_{L_0} \in A_\infty(\sigma)$ whenever $\omega_L \in A_\infty(\sigma)$.
- [Milakis-Pipher-Toro '13] For NTA domains with ADR boundary.
- [Cavero-Hofmann-Martell '18+ (with T. Toro '19)] For 1-sided NTA domains with ADR boundary [They proved more].
- See also the recent work of Mayboroda and Poggi.

# Approach II

**Theorem (R. Fefferman-Kenig-Pipher '91)**
Let $\Omega$ be a Lipschitz domain. If $\mathbf{a}^2(X)\frac{G_{L_0}(X)}{\delta^2(X)}\,dX$ satisfies the vanishing Carleson measure condition

$$\lim_{r\to 0}\sup_{\Delta\subseteq\partial\Omega}\left\{\frac{1}{\omega_{L_0}(\Delta)}\int_{T(\Delta)}\mathbf{a}^2(X)\frac{G_{L_0}(X)}{\delta^2(X)}\,dX\right\}^{1/2}=0$$

then $\omega_L \in \mathsf{RH}_2(\omega_{L_0})$.

# Approach II

**Theorem (R. Fefferman-Kenig-Pipher '91)**
Let $\Omega$ be a Lipschitz domain. If $\mathbf{a}^2(X)\frac{G_{L_0}(X)}{\delta^2(X)}\,dX$ satisfies the vanishing Carleson measure condition

$$\lim_{r\to 0}\sup_{\Delta\subseteq\partial\Omega}\left\{\frac{1}{\omega_{L_0}(\Delta)}\int_{T(\Delta)}\mathbf{a}^2(X)\frac{G_{L_0}(X)}{\delta^2(X)}\,dX\right\}^{1/2}=0$$

then $\omega_L \in \mathsf{RH}_2(\omega_{L_0})$.

- [Milakis-Pipher-Toro '13] For NTA domains with ADR boundary.

# Approach II

**Theorem (R. Fefferman-Kenig-Pipher '91)**
Let $\Omega$ be a Lipschitz domain. If $\mathbf{a}^2(X)\frac{G_{L_0}(X)}{\delta^2(X)}\,dX$ satisfies the vanishing Carleson measure condition

$$\lim_{r\to 0}\sup_{\Delta\subseteq\partial\Omega}\left\{\frac{1}{\omega_{L_0}(\Delta)}\int_{T(\Delta)}\mathbf{a}^2(X)\frac{G_{L_0}(X)}{\delta^2(X)}\,dX\right\}^{1/2}=0$$

then $\omega_L \in \mathsf{RH}_2(\omega_{L_0})$.

- [Milakis-Pipher-Toro '13] For NTA domains with ADR boundary.

**Question**
Can we go beyond NTA domains? Can we relax ADR of the boundary?

# Approach II

**Theorem (R. Fefferman-Kenig-Pipher '91)**
Let $\Omega$ be a Lipschitz domain. If $\mathbf{a}^2(X)\frac{G_{L_0}(X)}{\delta^2(X)}\,dX$ satisfies the vanishing Carleson measure condition

$$\lim_{r\to 0}\sup_{\Delta\subseteq\partial\Omega}\left\{\frac{1}{\omega_{L_0}(\Delta)}\int_{T(\Delta)}\mathbf{a}^2(X)\frac{G_{L_0}(X)}{\delta^2(X)}\,dX\right\}^{1/2}=0$$

then $\omega_L \in \mathrm{RH}_2(\omega_{L_0})$.

- [Milakis-Pipher-Toro '13] For NTA domains with ADR boundary.

**Question**
Can we go beyond NTA domains? Can we relax ADR of the boundary?

**Aim**: Study perturbation problem by replacing

1. NTA $\Rightarrow$ 1-sided NTA.
2. ADR $\Rightarrow$ Capacity Density Condition.

# Capacity density condition [uniform 2-fatness]

- Newtonian capacity of a compact set $E$ in a domain $D$ is defined by

$$\mathrm{Cap}(E, D) = \inf\left\{\int |\nabla v|^2 dX : \ v \in C_0^\infty(D),\ v(x) \geq \mathbf{1}_E\right\}.$$

- Capacity of a ball of radius $r$ is approximately $r^{n-1}$.

# Capacity density condition [uniform 2-fatness]

- Newtonian capacity of a compact set $E$ in a domain $D$ is defined by

$$\mathrm{Cap}(E,D) = \inf\left\{ \int |\nabla v|^2 \, dX : \; v \in C_0^\infty(D), \; v(x) \geq \mathbf{1}_E \right\}.$$

- Capacity of a ball of radius $r$ is approximately $r^{n-1}$.

- $\Omega$ satisfies the capacity density condition (CDC) if there is $c > 0$ s.t.

$$\frac{\mathrm{Cap}((\mathbb{R}^{n+1} \setminus \Omega) \cap \bar{B}(w,r), B(w,2r))}{\mathrm{Cap}(\bar{B}(w,r), B(w,2r))} \approx \frac{\mathrm{Cap}(\bar{B}(w,r) \setminus \Omega, B(w,2r))}{r^{n-1}} \geq c$$

for all $w \in \partial\Omega$ and $0 < r < \mathrm{diam}(\partial\Omega)$.

# Capacity density condition [uniform 2-fatness]

- Newtonian capacity of a compact set $E$ in a domain $D$ is defined by

$$\mathrm{Cap}(E, D) = \inf \left\{ \int |\nabla v|^2 \, dX : \ v \in C_0^\infty(D), \ v(x) \geq \mathbf{1}_E \right\}.$$

- Capacity of a ball of radius $r$ is approximately $r^{n-1}$.

- $\Omega$ satisfies the **capacity density condition** (CDC) if there is $c > 0$ s.t.

$$\frac{\mathrm{Cap}((\mathbb{R}^{n+1} \setminus \Omega) \cap \bar{B}(w,r), B(w,2r))}{\mathrm{Cap}(\bar{B}(w,r), B(w,2r))} \approx \frac{\mathrm{Cap}(\bar{B}(w,r) \setminus \Omega, B(w,2r))}{r^{n-1}} \geq c$$

for all $w \in \partial\Omega$ and $0 < r < \mathrm{diam}(\partial\Omega)$.

- CDC appeared first (?) in Pommerenki's work as a *uniformly perfect*.
Later used by Ancona, Aikawa, Lewis, Wannebo, Wu, Jones-Wolff, …

# CDC $\Rightarrow$ Wiener criterion

- CDC is a quantitative version of the Wiener criterion

$$\int_0^{r_0} \frac{\mathsf{Cap}((\mathbb{R}^n \setminus \Omega) \cap \bar{B}(w,r), B(w,2r))}{\mathsf{Cap}(\bar{B}(w,r), B(w,2r))} \frac{dr}{r} = \infty.$$

# CDC $\Rightarrow$ Wiener criterion

- CDC is a quantitative version of the Wiener criterion

$$\int_0^{r_0} \frac{\mathrm{Cap}((\mathbb{R}^n \setminus \Omega) \cap \bar{B}(w,r), B(w,2r))}{\mathrm{Cap}(\bar{B}(w,r), B(w,2r))} \frac{dr}{r} = \infty.$$

- [Wiener for Laplacian in '24 and Littman-Stampacchia-Weinberger '63] Wiener criterion is necessary and sufficient condition for a bounded domain to be regular for uniformly elliptic operators

# CDC $\Rightarrow$ Wiener criterion

- CDC is a quantitative version of the Wiener criterion

$$\int_0^{r_0} \frac{\mathrm{Cap}((\mathbb{R}^n \setminus \Omega) \cap \bar{B}(w,r), B(w,2r))}{\mathrm{Cap}(\bar{B}(w,r), B(w,2r))} \frac{dr}{r} = \infty.$$

- [Wiener for Laplacian in '24 and Littman-Stampacchia-Weinberger '63] Wiener criterion is necessary and sufficient condition for a bounded domain to be regular for uniformly elliptic operators

- [ CDC (Thickness) implies Fatness] If a closed set $\mathbb{R}^{n+1} \setminus \Omega$ is $\lambda$-thick for some $\lambda > n-1$, i.e., for all $0 < r \le r_0$ and $w \in \mathbb{R}^{n+1} \setminus \Omega$ with

$$\mathcal{H}^{\lambda}_{\infty}((\mathbb{R}^{n+1} \setminus \Omega) \cap \bar{B}(w,r)) \ge cr^{\lambda},$$

then $\Omega$ satisfies the CDC.

# CDC $\Rightarrow$ Wiener criterion

- CDC is a quantitative version of the Wiener criterion

$$\int_0^{r_0} \frac{\mathrm{Cap}((\mathbb{R}^n \setminus \Omega) \cap \bar{B}(w,r), B(w,2r))}{\mathrm{Cap}(\bar{B}(w,r), B(w,2r))} \frac{dr}{r} = \infty.$$

- [Wiener for Laplacian in '24 and Littman-Stampacchia-Weinberger '63] Wiener criterion is necessary and sufficient condition for a bounded domain to be regular for uniformly elliptic operators

- [ CDC (Thickness) implies Fatness] If a closed set $\mathbb{R}^{n+1} \setminus \Omega$ is $\lambda$-thick for some $\lambda > n-1$, i.e., for all $0 < r \leq r_0$ and $w \in \mathbb{R}^{n+1} \setminus \Omega$ with

$$\mathcal{H}^{\lambda}_{\infty}((\mathbb{R}^{n+1} \setminus \Omega) \cap \bar{B}(w,r)) \geq cr^{\lambda},$$

then $\Omega$ satisfies the CDC.

- Hence

$$\text{ADR} \implies \text{CDC} \implies \text{Wiener criterion.}$$

# First result

**Theorem (Akman-Hofmann-Martell-Toro '19)**

Let $\Omega$ be a 1-sided NTA domain satisfying the capacity density condition. Let

$$\|\mathbf{a}(A, A_0)\| := \sup_{B} \sup_{B'} \frac{1}{\omega_{L_0}^{X_\Delta}(\Delta')} \iint_{B' \cap \Omega} \mathbf{a}^2(X) \frac{G_{L_0}(X_\Delta, X)}{\delta(X)^2} \, dX,$$

where $\Delta = B \cap \partial\Omega$, $\Delta' = B' \cap \partial\Omega$, and the sups are taken respectively over all balls $B = B(x, r)$ with $x \in \partial\Omega$ and $0 < r < \mathrm{diam}(\partial\Omega)$, and $B' = B(x', r')$ with $x' \in 2\Delta$ and $0 < r' < rc_0/4$.

# First result

**Theorem (Akman-Hofmann-Martell-Toro '19)**
Let $\Omega$ be a 1-sided NTA domain satisfying the capacity density condition. Let

$$\|\!|\mathbf{a}(A, A_0)|\!\|:= \sup_B \sup_{B'} \frac{1}{\omega_{L_0}^{X_\Delta}(\Delta')} \iint_{B'\cap\Omega} \mathbf{a}^2(X) \frac{G_{L_0}(X_\Delta, X)}{\delta(X)^2}\, dX,$$

where $\Delta = B \cap \partial\Omega$, $\Delta' = B' \cap \partial\Omega$, and the sups are taken respectively over all balls $B = B(x, r)$ with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, and $B' = B(x', r')$ with $x' \in 2\Delta$ and $0 < r' < rc_0/4$.

(a) If $\|\!|\mathbf{a}(A, A_0)|\!\| < \infty$, then $\omega_L \in A_\infty(\omega_{L_0})$.

# First result

**Theorem (Akman-Hofmann-Martell-Toro '19)**
Let $\Omega$ be a 1-sided NTA domain satisfying the capacity density condition. Let

$$\|\!|\mathbf{a}(A, A_0)|\!\|:=\sup_{B}\sup_{B'}\frac{1}{\omega_{L_0}^{X_\Delta}(\Delta')}\iint_{B'\cap\Omega}\mathbf{a}^2(X)\frac{G_{L_0}(X_\Delta, X)}{\delta(X)^2}\,dX,$$

where $\Delta = B\cap\partial\Omega$, $\Delta' = B'\cap\partial\Omega$, and the sups are taken respectively over all balls $B = B(x, r)$ with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, and $B' = B(x', r')$ with $x' \in 2\Delta$ and $0 < r' < rc_0/4$.

(a) If $\|\!|\mathbf{a}(A, A_0)|\!\| < \infty$, then $\omega_L \in A_\infty(\omega_{L_0})$.

(b) Given $p$, $1 < p < \infty$, there exists $\epsilon_p$ such that if $\|\!|\mathbf{a}(A, A_0)|\!\| \le \varepsilon_p$, then $\omega_L \in \mathrm{RH}_p(\omega_{L_0})$.

# First result

**Theorem (Akman-Hofmann-Martell-Toro '19)**
Let $\Omega$ be a 1-sided NTA domain satisfying the capacity density condition. Let

$$\|\mathbf{a}(A, A_0)\| := \sup_{B} \sup_{B'} \frac{1}{\omega_{L_0}^{X_\Delta}(\Delta')} \iint_{B' \cap \Omega} \mathbf{a}^2(X) \frac{G_{L_0}(X_\Delta, X)}{\delta(X)^2} \, dX,$$

where $\Delta = B \cap \partial\Omega$, $\Delta' = B' \cap \partial\Omega$, and the sups are taken respectively over all balls $B = B(x, r)$ with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, and $B' = B(x', r')$ with $x' \in 2\Delta$ and $0 < r' < rc_0/4$.

(a) If $\|\mathbf{a}(A, A_0)\| < \infty$, then $\omega_L \in A_\infty(\omega_{L_0})$.

(b) Given $p$, $1 < p < \infty$, there exists $\epsilon_p$ such that if $\|\mathbf{a}(A, A_0)\| \leq \varepsilon_p$, then $\omega_L \in \mathrm{RH}_p(\omega_{L_0})$.

- If we assume $\partial\Omega$ is ADR then we can recover all the previous related results.

# First result

**Theorem (Akman-Hofmann-Martell-Toro '19)**
Let $\Omega$ be a 1-sided NTA domain satisfying the capacity density condition. Let

$$|||\mathbf{a}(A, A_0)||| := \sup_{B} \sup_{B'} \frac{1}{\omega_{L_0}^{X_\Delta}(\Delta')} \iint_{B' \cap \Omega} \mathbf{a}^2(X) \frac{G_{L_0}(X_\Delta, X)}{\delta(X)^2} \, dX,$$

where $\Delta = B \cap \partial\Omega$, $\Delta' = B' \cap \partial\Omega$, and the sups are taken respectively over all balls $B = B(x, r)$ with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, and $B' = B(x', r')$ with $x' \in 2\Delta$ and $0 < r' < rc_0/4$.

(a) If $|||\mathbf{a}(A, A_0)||| < \infty$, then $\omega_L \in A_\infty(\omega_{L_0})$.

(b) Given $p$, $1 < p < \infty$, there exists $\epsilon_p$ such that if $|||\mathbf{a}(A, A_0)||| \leq \varepsilon_p$, then $\omega_L \in \mathrm{RH}_p(\omega_{L_0})$.

- If we assume $\partial\Omega$ is ADR then we can recover all the previous related results.
- See recent work of Feneuil-Poggi who also obtained Part (a).

# Approach III

R. Fefferman considered

$$\mathcal{A}(\mathbf{a}(A, A_0))(x) := \left( \iint_{\Gamma(x)} \frac{\mathbf{a}^2(X)}{\delta^{n+1}(X)} \, dX \right)^{1/2}$$

where $\Gamma(x)$ is the non-tangential cone with vertex at $x \in \partial\Omega$.

# Approach III

R. Fefferman considered

$$\mathcal{A}(\mathbf{a}(A, A_0))(x) := \left( \iint_{\Gamma(x)} \frac{\mathbf{a}^2(X)}{\delta^{n+1}(X)} \, dX \right)^{1/2}$$

where $\Gamma(x)$ is the non-tangential cone with vertex at $x \in \partial\Omega$.

**Theorem (R. Fefferman '89 and R. Fefferman-Kenig-Pipher '91)**
Let $\Omega$ be a Lipschitz domain. If $\|\mathcal{A}(\mathbf{a}(A, A_0))\|_{L^\infty(\sigma)} \leq C_0 < \infty$ then $\omega_L \in A_\infty(\sigma)$ if $\omega_{L_0} \in A_\infty(\sigma)$.

# Approach III

R. Fefferman considered

$$\mathcal{A}(\mathbf{a}(A, A_0))(x) := \left( \iint_{\Gamma(x)} \frac{\mathbf{a}^2(X)}{\delta^{n+1}(X)} \, dX \right)^{1/2}$$

where $\Gamma(x)$ is the non-tangential cone with vertex at $x \in \partial\Omega$.

**Theorem (R. Fefferman '89 and R. Fefferman-Kenig-Pipher '91)**
Let $\Omega$ be a Lipschitz domain. If $\|\mathcal{A}(\mathbf{a}(A, A_0))\|_{L^\infty(\sigma)} \leq C_0 < \infty$ then $\omega_L \in A_\infty(\sigma)$ if $\omega_{L_0} \in A_\infty(\sigma)$.

This is sharp: $\|\mathcal{A}(\mathbf{a}(A, A_0))\|_{L^\infty(\sigma)} \leq C_0 \not\Rightarrow \omega_L \in \mathrm{RH}_p(\sigma)$ if $\omega_{L_0} \in \mathrm{RH}_p(\sigma)$.

# Approach III

R. Fefferman considered

$$\mathcal{A}(\mathbf{a}(A, A_0))(x) := \left( \iint_{\Gamma(x)} \frac{\mathbf{a}^2(X)}{\delta^{n+1}(X)} \, dX \right)^{1/2}$$

where $\Gamma(x)$ is the non-tangential cone with vertex at $x \in \partial\Omega$.

**Theorem (R. Fefferman '89 and R. Fefferman-Kenig-Pipher '91)**
Let $\Omega$ be a Lipschitz domain. If $\|\mathcal{A}(\mathbf{a}(A, A_0))\|_{L^\infty(\sigma)} \leq C_0 < \infty$ then $\omega_L \in A_\infty(\sigma)$ if $\omega_{L_0} \in A_\infty(\sigma)$.

This is sharp: $\|\mathcal{A}(\mathbf{a}(A, A_0))\|_{L^\infty(\sigma)} \leq C_0 \not\Rightarrow \omega_L \in \mathrm{RH}_p(\sigma)$ if $\omega_{L_0} \in \mathrm{RH}_p(\sigma)$.

- [R. Fefferman-Kenig-Pipher '91] Using "$L^r$-averages" of the disagreement function $\mathbf{a}$ and assuming it satisfies a certain vanishing Carleson measure condition then the exponent $p$ can be preserved.

# Approach III

R. Fefferman considered

$$\mathcal{A}(\mathbf{a}(A, A_0))(x) := \left( \iint_{\Gamma(x)} \frac{\mathbf{a}^2(X)}{\delta^{n+1}(X)}\, dX \right)^{1/2}$$

where $\Gamma(x)$ is the non-tangential cone with vertex at $x \in \partial\Omega$.

**Theorem (R. Fefferman '89 and R. Fefferman-Kenig-Pipher '91)**
Let $\Omega$ be a Lipschitz domain. If $\|\mathcal{A}(\mathbf{a}(A, A_0))\|_{L^\infty(\sigma)} \leq C_0 < \infty$ then $\omega_L \in A_\infty(\sigma)$ if $\omega_{L_0} \in A_\infty(\sigma)$.

This is sharp: $\|\mathcal{A}(\mathbf{a}(A, A_0))\|_{L^\infty(\sigma)} \leq C_0 \not\Rightarrow \omega_L \in \mathrm{RH}_p(\sigma)$ if $\omega_{L_0} \in \mathrm{RH}_p(\sigma)$.

- [R. Fefferman-Kenig-Pipher '91] Using "$L^r$-averages" of the disagreement function $\mathbf{a}$ and assuming it satisfies a certain vanishing Carleson measure condition then the exponent $p$ can be preserved.
- [Milakis-Pipher-Toro '13] For NTA domains with ADR boundary.

# Second result

Theorem (Akman-Hofmann-Martell-Toro '19)
Let $\Omega$ be a 1-sided NTA domain satisfying the capacity density condition. Given $\alpha > 0$, set

$$\mathcal{A}_\alpha(\mathbf{a}(A, A_0))(x) := \left( \iint_{\Gamma_\alpha(x)} \frac{\mathbf{a}(X)^2}{\delta(X)^{n+1}} \, dX \right)^{\frac{1}{2}}, \qquad x \in \partial\Omega,$$

where $\Gamma_\alpha(x) = \{ Y \in \Omega : |Y - x| < (1+\alpha)\delta(Y) \}$.

# Second result

**Theorem (Akman-Hofmann-Martell-Toro '19)**
Let $\Omega$ be a 1-sided NTA domain satisfying the capacity density condition. Given $\alpha > 0$, set

$$\mathcal{A}_\alpha(\mathbf{a}(A, A_0))(x) := \left( \iint_{\Gamma_\alpha(x)} \frac{\mathbf{a}(X)^2}{\delta(X)^{n+1}} \, dX \right)^{\frac{1}{2}}, \qquad x \in \partial\Omega,$$

where $\Gamma_\alpha(x) = \{ Y \in \Omega : |Y - x| < (1+\alpha)\delta(Y) \}$.

(i) If $\mathcal{A}_\alpha(\mathbf{a}(A, A_0)) \in L^\infty(\omega_{L_0})$, then $\omega_L \in A_\infty(\omega_{L_0})$.

# Second result

**Theorem (Akman-Hofmann-Martell-Toro '19)**
Let $\Omega$ be a 1-sided NTA domain satisfying the capacity density condition. Given $\alpha > 0$, set

$$\mathcal{A}_\alpha(\mathbf{a}(A, A_0))(x) := \left( \iint_{\Gamma_\alpha(x)} \frac{\mathbf{a}(X)^2}{\delta(X)^{n+1}}\, dX \right)^{\frac{1}{2}}, \qquad x \in \partial\Omega,$$

where $\Gamma_\alpha(x) = \{ Y \in \Omega : |Y - x| < (1+\alpha)\delta(Y) \}$.

(i) If $\mathcal{A}_\alpha(\mathbf{a}(A, A_0)) \in L^\infty(\omega_{L_0})$, then $\omega_L \in A_\infty(\omega_{L_0})$.

(ii) Given $p$, $1 < p < \infty$, there exists $\varepsilon_p > 0$, such that if $\mathcal{A}_\alpha(\mathbf{a}(A, A_0)) \in L^\infty(\omega_{L_0})$ with $\|\mathcal{A}_\alpha(\mathbf{a}(A, A_0))\|_{L^\infty(\omega_{L_0})} \le \varepsilon_p$, then $\omega_L \in \mathrm{RH}_p(\omega_{L_0})$.

# Second result

**Theorem (Akman-Hofmann-Martell-Toro '19)**
Let $\Omega$ be a 1-sided NTA domain satisfying the capacity density condition. Given $\alpha > 0$, set

$$\mathcal{A}_\alpha(\mathbf{a}(A, A_0))(x) := \left( \iint_{\Gamma_\alpha(x)} \frac{\mathbf{a}(X)^2}{\delta(X)^{n+1}} \, dX \right)^{\frac{1}{2}}, \qquad x \in \partial\Omega,$$

where $\Gamma_\alpha(x) = \{ Y \in \Omega : |Y - x| < (1+\alpha)\delta(Y) \}$.

(i) If $\mathcal{A}_\alpha(\mathbf{a}(A, A_0)) \in L^\infty(\omega_{L_0})$, then $\omega_L \in A_\infty(\omega_{L_0})$.

(ii) Given $p$, $1 < p < \infty$, there exists $\varepsilon_p > 0$, such that if $\mathcal{A}_\alpha(\mathbf{a}(A, A_0)) \in L^\infty(\omega_{L_0})$ with $\|\mathcal{A}_\alpha(\mathbf{a}(A, A_0))\|_{L^\infty(\omega_{L_0})} \leq \varepsilon_p$, then $\omega_L \in \mathrm{RH}_p(\omega_{L_0})$.

Part (ii) is new even in the case of nice domains such as the unit ball, the upper-half space, or non-tangentially accessible domains.

- If we assume $\partial\Omega$ is ADR then we can recover all the previous related results.

# Sketch of the proof of second result

- It can be shown that

$$\frac{C}{\omega_0(\Delta(x,r))}\iint_{T(\Delta(x,r))}\mathbf{a}^2(X)\frac{G_0(X)}{\delta^2(X)}\,dX \le \frac{1}{\omega_0(\Delta)}\int_{\Delta(x,2r)}(\mathcal{A}(\mathbf{a})(x))^2\,d\omega_0(x).$$

Hence $|||\mathbf{a}(A,A_0)||| \lesssim_\alpha \|\mathcal{A}_\alpha(\mathbf{a}(A,A_0))\|^2_{L^\infty(\omega_{L_0})}$.

Our first result implies the desired conclusions in the second result.

# Sketch of the proof of second result

- It can be shown that

$$\frac{C}{\omega_0(\Delta(x,r))}\iint_{T(\Delta(x,r))}\mathbf{a}^2(X)\frac{G_0(X)}{\delta^2(X)}\,dX \le \frac{1}{\omega_0(\Delta)}\int_{\Delta(x,2r)}(\mathcal{A}(\mathbf{a})(x))^2\,d\omega_0(x).$$

Hence $|||\mathbf{a}(A,A_0)||| \lesssim_\alpha \|\mathcal{A}_\alpha(\mathbf{a}(A,A_0))\|^2_{L^\infty(\omega_{L_0})}$.

Our first result implies the desired conclusions in the second result.

- **A key property:** One needs to know elliptic measure is doubling and Sawtooth domains inherit the good properties of the original domains. That is, in this regime, all proper sawtooth domains obtained from the dyadic cubes of $\Omega$ should be a 1-sided NTA domains satisfying Capacity density condition.

# A key identity

- **A key identity:** Let $\Omega$ be a bounded 1-sided NTA domain satisfying the CDC, and let $L_0 = -\operatorname{div}(A_0\nabla)$ and $L = -\operatorname{div}(A\nabla)$ be two real (non-necessarily symmetric) uniformly elliptic operators. Given $g \in \text{Lip}(\partial\Omega)$, consider the solutions $u_L$ and $u_{L_0}$ given by

$$u_{L_0}(X) = \int_{\partial\Omega} g(y)\, d\omega_{L_0}^X(y), \qquad u_L(X) = \int_{\partial\Omega} g(y)\, d\omega_L^X(y), \qquad X \in \Omega.$$

Then,

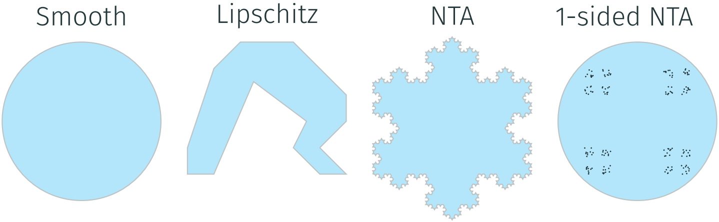
$$u_L(X) - u_{L_0}(X) = \iint_\Omega (A_0 - A)^\top(Y) \nabla_Y G_{L^\top}(Y, X) \cdot \nabla u_{L_0}(Y)\, dY$$

for almost every $X \in \Omega$.

# Some future problems

**Question**
*Can we go beyond 1-sided NTA setting (assuming the CDC)?*

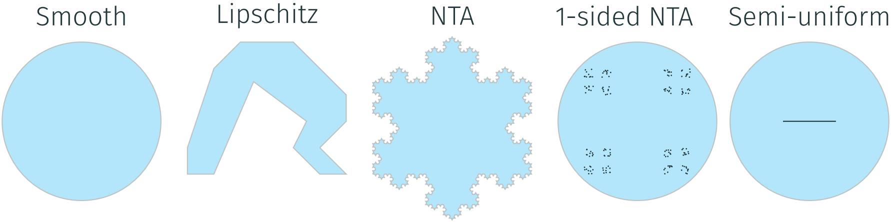
$$\textit{Smooth} \subsetneq \textit{Lipschitz} \subsetneq \textit{NTA} \subsetneq \textit{1-sided NTA} \subsetneq \textit{Semi-uniform}$$



# Some future problems

**Question**
*Can we go beyond 1-sided NTA setting (assuming the CDC)?*

$$\textit{Smooth} \subsetneq \textit{Lipschitz} \subsetneq \textit{NTA} \subsetneq \textit{1-sided NTA} \subsetneq \textit{Semi-uniform}$$

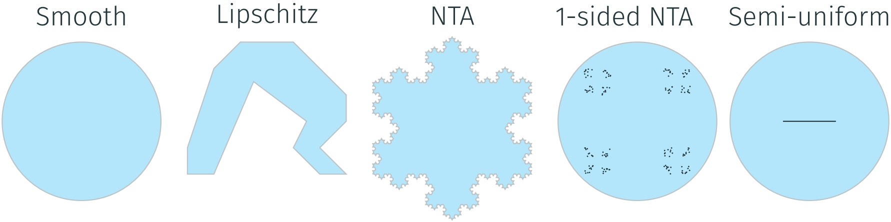



- A domain $\Omega$ is called **semi-uniform** if every $x \in \Omega$ and $y \in \partial\Omega$ can be connected by a cigar curve $\gamma$ such that $\gamma \setminus \{y\} \subset \Omega$ and length$(\gamma) \leq C|x - y|$.

# Some future problems

**Question**
*Can we go beyond 1-sided NTA setting (assuming the CDC)?*

$$\textit{Smooth} \subsetneq \textit{Lipschitz} \subsetneq \textit{NTA} \subsetneq \textit{1-sided NTA} \subsetneq \textit{Semi-uniform}$$




- A domain $\Omega$ is called **semi-uniform** if every $x \in \Omega$ and $y \in \partial\Omega$ can be connected by a cigar curve $\gamma$ such that $\gamma \setminus \{y\} \subset \Omega$ and length$(\gamma) \leq C|x - y|$.
- [Aikawa and Hirata '08, Azzam '18] If $\Omega$ is a semi-uniform domain satisfying the CDC then its harmonic measure is doubling.

# Sketch of the proof

Localize the problem. It is enough to show that:

Let $\Omega$ be a 1-sided NTA domain satisfying the capacity density condition. Let $\mathbf{a}(X) := \sup_{Y \in B(X,\delta(X)/2)} |A(Y) - A_0(Y)|$ where $\delta(X) := \operatorname{dist}(X, \partial\Omega)$. Fix $x_0 \in \partial\Omega$, and let $B_0 = B(x_0, r_0)$, $0 < r_0 < \operatorname{diam}(\partial\Omega)$, and $\Delta_0 = B_0 \cap \partial\Omega$. Let

$$\|\mathbf{a}(A, A_0)\|_{B_0} := \sup_B \frac{1}{\omega_{L_0}^{X_{\Delta_0}}(\Delta)} \iint_{B\cap\Omega} \mathbf{a}^2(X) \frac{G_{L_0}(X_{\Delta_0}, X)}{\delta(X)^2} \, dX,$$

where $\Delta = B \cap \partial\Omega$ and the supremum is taken over all balls $B = B(x, r)$ with $x \in 2\Delta_0$ and $0 < r < r_0 c_0/4$, and $c_0$ is the Corkscrew constant.

(a) If $\|\mathbf{a}(A, A_0)\| < \infty$, then $\omega_L \in A_\infty(\Delta_0, \omega_{L_0})$.

(b) Given $p$, $1 < p < \infty$, there exists $\epsilon_p$ such that if $\|\mathbf{a}(A, A_0)\| \leq \varepsilon_p$, then $\omega_L \in \mathrm{RH}_p(\Delta_0, \omega_{L_0})$.

# Sketch of the proof

For fixed $j \in \mathbb{N}$, let

$$\tilde{A}_j(Y) := \begin{cases} A(Y) & \text{if } \delta(Y) \geq 2^{-j}, \\ A_0(Y) & \text{if } \delta(Y) < 2^{-j}. \end{cases}$$

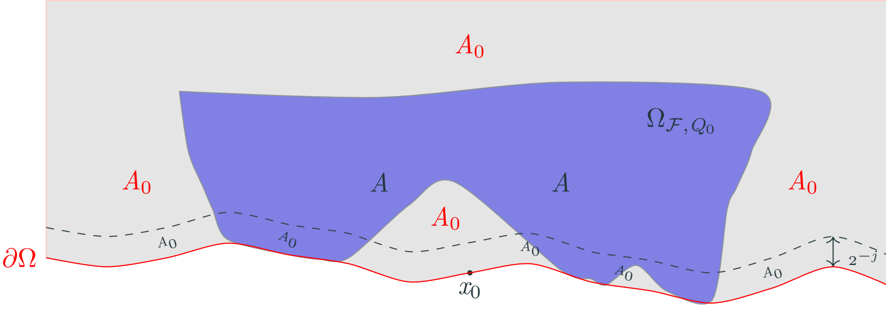
Aim: If $\tilde{L}_j$ is the operator associated to $\tilde{L}_j u = -\operatorname{div}(\tilde{A}_j u)$ and $\omega_{\tilde{L}_j}$ is the elliptic measure of $\Omega$ associated to operator $\tilde{L}_j$ then it is enough to show that $\omega_{\tilde{L}_j} \in A_\infty(\frac{5}{4}\Delta_0, \omega_{L_0})$ for part (a) and for part (b) enough to show that $\omega_{\tilde{L}_j} \in \mathrm{RH}_p(\frac{5}{4}\Delta_0, \omega_{L_0})$ with uniform constants depending only on the allowable parameters.

# Step 1

To this end, we further modify our operator $\tilde{L}_j$ in $Q_0$, consider

$$\bar{A}(Y) := \begin{cases} \tilde{A}_j(Y) & \text{if } Y \in \Omega_{\mathcal{F},Q_0}, \\ A_0(Y) & \text{if } Y \in \Omega \setminus \Omega_{\mathcal{F},Q_0}. \end{cases}$$

where $\tilde{A}_j(Y)$ is as above for fixed $j$, $T(\Delta(x_0, r_0))$ is the Carleson region associated to $Q_0$ and $Q_0$ is also as above.

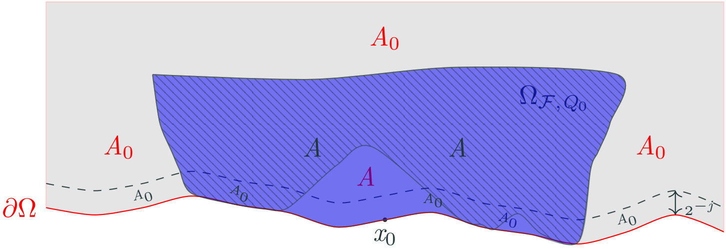



For this operator, we prove $\bar{\omega} \in \mathrm{RH}_p(\omega_{L_0})$.

## Step 2

We now change the operator $\bar{A}$ in the Carleson region to complete the process.

$$\hat{A}(Y) := \begin{cases} A_1(Y) & \text{if } Y \in \Omega \setminus (T_{Q_0} \setminus \Omega_{\mathcal{F},Q_0}), \\ \tilde{A}_j(Y) & \text{if } Y \in T_{Q_0} \setminus \Omega_{\mathcal{F},Q_0}. \end{cases}$$



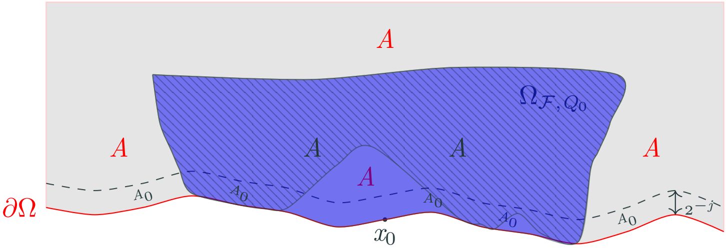

For this operator, we prove $\hat{\omega} \in \mathrm{RH}_p(\omega_{L_0})$.

# Step 3:

In this part, we change the operator outside of $T_{Q_0}$ to complete the process. To this end, let $L_3 u = -\operatorname{div}(A_3 \nabla u)$, where

$$A_3(Y) := \begin{cases} A_2(Y) & \text{if } Y \in T_{Q_0}, \\ \widetilde{A}(Y) & \text{if } Y \in \Omega \setminus T_{Q_0}, \end{cases}$$

and note that $L_3 \equiv \widetilde{L}$ in $\Omega$. Let $w_3^{X_0} := \omega_{L_3}^{X_0}$ be the elliptic measure of $\Omega$ associated with the operator $L_3 \equiv \widetilde{L}$ with pole at $X_0$.




For this operator, we prove $\omega_3 \in \mathrm{RH}_p(\omega_{L_0})$.

Thank you for your attention!!!

**Temporary page!**

LaTeX was unable to guess the total number of pages correctly. As there was some unprocessed data that should have been added t the final page this extra page has been added to receive it.

If you rerun the document (without altering it) this surplus page w go away, because LaTeX now knows how many pages to expect for th document.