On the logarithm of the minimizing integrand for certain variational problems in two dimensions

Murat Akman John L. Lewis Andrew Vogel

University of Kentucky University of Kentucky Syracuse University

2012 Spring Central Section Meeting
University of Kansas, Lawrence, KS
March 31, 2012

UK UNIVERSITY OF KENTUCKY

Outline

Outline

Let Ω denote a bounded region in the complex plane $\mathbb{C}$. Given p, $1 < p < \infty$, let $z = x_1 + ix_2$ denote points in $\mathbb{C}$ and let $W^{1,p}(\Omega)$ denote equivalence classes of functions $h : \mathbb{C} \to \mathbb{R}$ with distributional gradient $\nabla h = h_{x_1} + ih_{x_2}$ and Sobolev norm

$$\|h\|_{W^{1,p}(\Omega)} = (\int_{\Omega} (|h|^p + |\nabla h|^p) dA)^{1/p} < \infty$$

where dA denotes two dimensional Lebesgue measure.

Let Ω denote a bounded region in the complex plane $\mathbb{C}$. Given p, $1 < p < \infty$, let $z = x_1 + ix_2$ denote points in $\mathbb{C}$ and let $W^{1,p}(\Omega)$ denote equivalence classes of functions $h : \mathbb{C} \to \mathbb{R}$ with distributional gradient $\nabla h = h_{x_1} + ih_{x_2}$ and Sobolev norm

$$\|h\|_{W^{1,p}(\Omega)} = (\int_{\Omega} (|h|^p + |\nabla h|^p) dA)^{1/p} < \infty$$

where dA denotes two dimensional Lebesgue measure.

Let $C_0^\infty(\Omega)$ denote infinitely differentiable functions with compact support in Ω and let $W_0^{1,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,p}(\Omega)$.

Let Ω denote a bounded region in the complex plane $\mathbb{C}$. Given p, $1 < p < \infty$, let $z = x_1 + ix_2$ denote points in $\mathbb{C}$ and let $W^{1,p}(\Omega)$ denote equivalence classes of functions $h : \mathbb{C} \to \mathbb{R}$ with distributional gradient $\nabla h = h_{x_1} + ih_{x_2}$ and Sobolev norm

$$\|h\|_{W^{1,p}(\Omega)} = \Big(\int_\Omega (|h|^p + |\nabla h|^p) dA\Big)^{1/p} < \infty$$

where dA denotes two dimensional Lebesgue measure.
Let $C_0^\infty(\Omega)$ denote infinitely differentiable functions with compact support in Ω and let $W_0^{1,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,p}(\Omega)$.
Let $f : \mathbb{C} \setminus \{0\} \to (0, \infty)$ be homogeneous of degree p on $\mathbb{C} \setminus \{0\}$. That is,

$$f(\eta) = |\eta|^p f\Big(\frac{\eta}{|\eta|}\Big) > 0 \text{ when } \eta \in \mathbb{C} \setminus \{0\}.$$

Let Ω denote a bounded region in the complex plane $\mathbb{C}$. Given p, $1 < p < \infty$, let $z = x_1 + ix_2$ denote points in $\mathbb{C}$ and let $W^{1,p}(\Omega)$ denote equivalence classes of functions $h : \mathbb{C} \to \mathbb{R}$ with distributional gradient $\nabla h = h_{x_1} + ih_{x_2}$ and Sobolev norm

$$\|h\|_{W^{1,p}(\Omega)} = (\int_{\Omega} (|h|^p + |\nabla h|^p) dA)^{1/p} < \infty$$

where dA denotes two dimensional Lebesgue measure.
Let $C_0^\infty(\Omega)$ denote infinitely differentiable functions with compact support in Ω and let $W_0^{1,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,p}(\Omega)$.
Let $f : \mathbb{C} \setminus \{0\} \to (0, \infty)$ be homogeneous of degree p on $\mathbb{C} \setminus \{0\}$. That is,

$$f(\eta) = |\eta|^p f(\frac{\eta}{|\eta|}) > 0 \ \text{ when } \ \eta \in \mathbb{C} \setminus \{0\}.$$

Assume also that f is strictly convex in $\mathbb{C} \setminus \{0\}$.

Given $h \in W^{1,p}(\Omega)$, let $E = \{h + \phi : \phi \in W_0^{1,p}(\Omega)\}$. It is well known **(see Heinonen, Kilpelainen, Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Chapter 5, Dover Publications, 2006)** that Euler equation has unique minimizer $u \in E$, i.e

$$\inf_{w \in E} \int_{\Omega} f(\nabla w)\, dA = \int_{\Omega} f(\nabla u)\, dA \ \text{ for some } \ u \in E.$$

Given $h \in W^{1,p}(\Omega)$, let $E = \{h + \phi : \phi \in W_0^{1,p}(\Omega)\}$. It is well known **(see Heinonen, Kilpelainen, Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Chapter 5, Dover Publications, 2006)** that Euler equation has unique minimizer $u \in E$, i.e

$$\inf_{w \in E} \int_\Omega f(\nabla w) dA = \int_\Omega f(\nabla u) dA \text{ for some } u \in E.$$

Moreover, u is a weak solution at $z \in \Omega$ to the Euler equation

$$0 = \nabla \cdot (\nabla f(\nabla u(z))) = \sum_{k,j=1}^{2} f_{\eta_k \eta_j}(\nabla u(z))\, u_{x_k x_j}(z). \tag{1}$$

Given $h \in W^{1,p}(\Omega)$, let $E = \{h + \phi : \phi \in W_0^{1,p}(\Omega)\}$. It is well known **(see Heinonen, Kilpelainen, Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Chapter 5, Dover Publications, 2006)** that Euler equation has unique minimizer $u \in E$, i.e

$$\inf_{w \in E} \int_\Omega f(\nabla w)dA = \int_\Omega f(\nabla u)dA \text{ for some } u \in E.$$

Moreover, u is a weak solution at $z \in \Omega$ to the Euler equation

$$0 = \nabla \cdot (\nabla f(\nabla u(z))) = \sum_{k,j=1}^{2} f_{\eta_k \eta_j}(\nabla u(z))\, u_{x_k x_j}(z). \tag{1}$$

That is,

$$\int_\Omega \langle \nabla f(\nabla u(z)), \nabla \theta(z) \rangle dA = 0 \text{ whenever } \theta \in W_0^{1,p}(\Omega).$$

Here $\nabla\cdot$ denotes divergence in the $z = x_1 + ix_2$ variable and $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{C}$.

Here $\nabla\cdot$ denotes divergence in the $z = x_1 + ix_2$ variable and $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{C}$.
Moreover, if f is sufficiently 'smooth', it follows from either Schauder theory or the fact that ∇u is a quasiregular mapping of $\mathbb{C}$ that u has continuous third derivatives in a neighborhood of z whenever $\nabla u(z) \neq 0$.

Here $\nabla\cdot$ denotes divergence in the $z = x_1 + ix_2$ variable and $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{C}$.
Moreover, if f is sufficiently 'smooth', it follows from either Schauder theory or the fact that ∇u is a quasiregular mapping of $\mathbb{C}$ that u has continuous third derivatives in a neighborhood of z whenever $\nabla u(z) \neq 0$. In this case, (1) holds pointwise and we can differentiate this equation with respect to x_l, $l = 1, 2$ to get

$$0 = \nabla \cdot \left(\frac{\partial}{\partial x_l} (\nabla f(\nabla u(z))) \right) = \sum_{k,j=1}^{2} \frac{\partial}{\partial x_k} \left(\frac{\partial^2 f}{\partial \eta_k \eta_j} (\nabla u(z))\, u_{x_j x_l} \right)$$

Here $\nabla\cdot$ denotes divergence in the $z = x_1 + ix_2$ variable and $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{C}$.
Moreover, if f is sufficiently 'smooth', it follows from either Schauder theory or the fact that ∇u is a quasiregular mapping of $\mathbb{C}$ that u has continuous third derivatives in a neighborhood of z whenever $\nabla u(z) \neq 0$. In this case, (1) holds pointwise and we can differentiate this equation with respect to x_l, $l = 1, 2$ to get

$$0 = \nabla \cdot \left(\frac{\partial}{\partial x_l} (\nabla f(\nabla u(z))) \right) = \sum_{k,j=1}^{2} \frac{\partial}{\partial x_k} \left(\frac{\partial^2 f}{\partial \eta_k \eta_j} (\nabla u(z))\, u_{x_j x_l} \right)$$

From this display we see that if $\nabla u(z) \neq 0$, and u, f are sufficiently smooth, then $\zeta = u_{x_l}$ satisfies

$$L\zeta = \sum_{k,j=1}^{2} \frac{\partial}{\partial x_k} \left(b_{kj}(z) \frac{\partial \zeta}{\partial x_j} \right) = 0 \qquad (2)$$

where $b_{kj}(z) = f_{\eta_k \eta_j}(\nabla u(z))$ when $1 \leq k, j \leq 2$.

Using the homogeneity of f and Euler's formula for $k = 1, 2$ and if $\eta \neq 0$, then

$$\sum_{j=1}^{2} \eta_j \, f_{\eta_k \eta_j}(\eta) = (p-1) f_{\eta_k}(\eta) \quad \text{and} \quad \sum_{k=1}^{2} \eta_k \, f_{\eta_k}(\eta) = p f(\eta). \tag{3}$$

Using the homogeneity of f and Euler's formula for $k = 1, 2$ and if $\eta \neq 0$, then

$$\sum_{j=1}^{2} \eta_j \, f_{\eta_k \eta_j}(\eta) = (p-1) f_{\eta_k}(\eta) \ \text{ and } \ \sum_{k=1}^{2} \eta_k \, f_{\eta_k}(\eta) = p f(\eta). \qquad (3)$$

Putting u in for ζ in (2) and using (3) and (1), it follows that

$$Lu = \sum_{k,j=1}^{2} \frac{\partial}{\partial x_k} \left(f_{\eta_k \eta_j}(\nabla u(z)) \frac{\partial u}{\partial x_j} \right) = (p-1) \sum_{k=1}^{2} \frac{\partial}{\partial x_k} \left(f_{\eta_k}(\nabla u(z)) \right) = 0.$$

Using the homogeneity of f and Euler's formula for $k = 1, 2$ and if $\eta \neq 0$, then

$$\sum_{j=1}^{2} \eta_j \, f_{\eta_k \eta_j}(\eta) = (p-1) f_{\eta_k}(\eta) \text{ and } \sum_{k=1}^{2} \eta_k \, f_{\eta_k}(\eta) = p f(\eta). \tag{3}$$

Putting u in for ζ in (2) and using (3) and (1), it follows that

$$Lu = \sum_{k,j=1}^{2} \frac{\partial}{\partial x_k} \left(f_{\eta_k \eta_j}(\nabla u(z)) \frac{\partial u}{\partial x_j} \right) = (p-1) \sum_{k=1}^{2} \frac{\partial}{\partial x_k} \left(f_{\eta_k}(\nabla u(z)) \right) = 0.$$

Hence, $\zeta = u$ is also solution to $L\zeta = 0$ in a neighborhood of z.

Outline

Using

$$L\zeta = \sum_{k,j=1}^{2} \frac{\partial}{\partial x_k}\left(b_{kj}(z)\frac{\partial \zeta}{\partial x_j}\right) = 0$$

for $\zeta = u_{x_l}$, $l = 1, 2$ and $\zeta = u$

Using

$$L\zeta = \sum_{k,j=1}^{2} \frac{\partial}{\partial x_k}\left(b_{kj}(z) \frac{\partial \zeta}{\partial x_j} \right) = 0$$

for $\zeta = u_{x_l}$, $l = 1, 2$ and $\zeta = u$ we prove

Theorem (Akman, Lewis, Vogel)

In a neighborhood of z and under the above assumptions, $\log f(\nabla u))$ is a sub solution, solution, or super solution to L when $p > 2$, $p = 2$, $p < 2$, respectively.

Let $B(z,r) = \{w \in \mathbb{C} : |w - z| < r\}$ whenever $z \in \mathbb{C}$ and $r > 0$.
Let $d(E,F)$ denote the distance between the sets $E, F \subset \mathbb{C}$.

Let $B(z,r) = \{w \in \mathbb{C} : |w - z| < r\}$ whenever $z \in \mathbb{C}$ and $r > 0$.
Let $d(E,F)$ denote the distance between the sets $E, F \subset \mathbb{C}$.
Let $\lambda > 0$ be a positive function on $(0, r_0)$ with $\lim_{r \longrightarrow 0} \lambda(r) = 0$.
Define H^λ Hausdorff measure on $\mathbb{C}$ as follows:
For fixed $0 < \delta < r_0$ and $E \subseteq \mathbb{C}$, let $L(\delta) = \{B(z_i, r_i)\}$ be such that $E \subseteq \bigcup B(z_i, r_i)$ and $0 < r_i < \delta, \; i = 1, 2, \ldots$ Set

$$\phi_\delta^\lambda(E) = \inf_{L(\delta)} \sum \lambda(r_i).$$

Let $B(z,r) = \{w \in \mathbb{C} : |w - z| < r\}$ whenever $z \in \mathbb{C}$ and $r > 0$.
Let $d(E,F)$ denote the distance between the sets $E, F \subset \mathbb{C}$.
Let $\lambda > 0$ be a positive function on $(0, r_0)$ with $\lim\limits_{r \to 0} \lambda(r) = 0$.
Define H^λ Hausdorff measure on $\mathbb{C}$ as follows:
For fixed $0 < \delta < r_0$ and $E \subseteq \mathbb{C}$, let $L(\delta) = \{B(z_i, r_i)\}$ be such that $E \subseteq \bigcup B(z_i, r_i)$ and $0 < r_i < \delta$, $i = 1, 2, \ldots$ Set

$$\phi_\delta^\lambda(E) = \inf_{L(\delta)} \sum \lambda(r_i).$$

Then

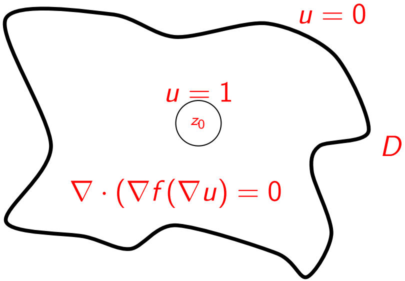
$$H^\lambda(E) = \lim_{\delta \to 0} \phi_\delta^\lambda(E).$$

In case $\lambda(r) = r^\alpha$ we write H^α for H^λ.

Next, suppose $D \subset \mathbb{C}$ is a bounded simply connected domain, $z_o \in D, \Omega = D \setminus B(z_0, \frac{1}{2}d(z_0, \partial D))$, and u is minimizer for variational problem in Ω

$$\inf \int\limits_{\Omega} f(\nabla w) dA = \int\limits_{\Omega} f(\nabla u) dA$$

with boundary values $u = 1$ on $\partial B(z_0, \frac{1}{2}d(z_0, \partial D))$ and $u = 0$ on ∂D in the $W^{1,p}(\Omega)$ sense. Put $u \equiv 0$ outside of D.



Then it follows from **(see Heinonen, Kilpelainen, Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Chapter 21, Dover Publications, 2006)** that there exists a unique finite positive Borel measure μ with support on ∂D satisfying

$$\int_{\mathbb{C}} \langle \nabla f(\nabla u(z)), \nabla\theta(z) \rangle dA = -\int \theta d\mu \qquad (*)$$

whenever $\theta \in C_0^\infty(\mathbb{C} \setminus \overline{B}(z_0, \frac{1}{2}d(z_0, \partial D)))$.

Then it follows from **(see Heinonen, Kilpelainen, Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Chapter 21, Dover Publications, 2006)** that there exists a unique finite positive Borel measure μ with support on ∂D satisfying

$$\int_{\mathbb{C}} \langle \nabla f(\nabla u(z)), \nabla \theta(z) \rangle dA = -\int \theta d\mu \qquad (*)$$

whenever $\theta \in C_0^\infty(\mathbb{C} \setminus \overline{B}(z_0, \frac{1}{2} d(z_0, \partial D)))$.
Define the Hausdorff dimension of μ denoted H-dim μ, by

H-dim $\mu = \inf\{\alpha : \exists E \text{ Borel } \subset \partial\Omega \text{ with } H^\alpha(E) = 0 \text{ and } \mu(E) = \mu(\partial\Omega)\}$.

Outline

If $f(\nabla u) = |\nabla u|^2$, i.e, when u is harmonic, and $\mu = \omega$ is harmonic measure with respect to a point in D. In [M], Makarov proved that

Theorem (Makarov)

a) *μ is concentrated on a set of σ finite H^1 measure.*
b) *There exists $0 < A < \infty$, such that μ is absolutely continuous with respect to Hausdorff measure defined relative to λ where*

$$\lambda(r) = r\,\exp[A\sqrt{\log\frac{1}{r}\,\log\log\log\frac{1}{r}}],\quad 0 < r < 10^{-6}.$$

If $f(\nabla u) = |\nabla u|^2$, i.e, when u is harmonic, and $\mu = \omega$ is harmonic measure with respect to a point in D. In [M], Makarov proved that

Theorem (Makarov)

a) *μ is concentrated on a set of σ finite H^1 measure.*

b) *There exists $0 < A < \infty$, such that μ is absolutely continuous with respect to Hausdorff measure defined relative to λ where*

$$\lambda(r) = r \exp[A\sqrt{\log \frac{1}{r} \log\log\log \frac{1}{r}}], \quad 0 < r < 10^{-6}.$$

In [BL], Bennewitz and Lewis have attempted to generalize this result for μ defined as in $(*)$ relative to $f(\nabla u) = |\nabla u|^p$.

Theorem (Bennewitz, Lewis)

If $\partial\Omega$ is a quasicircle, then H-dim $\mu \le 1$ for $2 < p < \infty$ while H-dim $\mu \ge 1$ for $1 < p < 2$. Moreover, if $\partial\Omega$ is the von Koch snowflake then strict inequality holds for H-dim μ.

In [LNP], Lewis, Nyström, and Poggi-Corradini proved that

Theorem (Lewis, Nyström, Poggi-Corradini)

Let $D \subset \mathbb{C}$ be a bounded simply connected domain and $1 < p < \infty$, $p \neq 2$. Put

$$\lambda(r) = r \exp[A\sqrt{\log \frac{1}{r} \log\log \frac{1}{r}}],\ 0 < r < 10^{-6}.$$

Then

a) *If $p > 2$, there exists $A = A(p) \leq -1$ such that μ is concentrated on a set of σ finite H^{λ} measure.*
b) *If $1 < p < 2$, there exists $A = A(p) \geq 1$, such that μ is absolutely continuous with respect to H^{λ}.*

Outline

Makarov's result corresponds to $f(\eta) = |\eta|^2$, which is homogeneous of degree 2. In the future, we want to generalize this result for f which is homogeneous of degree 2 on $\mathbb{C} \setminus \{0\}$ and strictly convex on $\mathbb{C} \setminus \{0\}$ and the measure μ relative to f defined as

$$\int_{\mathbb{C}} \langle \nabla f(\nabla u(z)), \nabla \theta(z) \rangle dA = - \int \theta d\mu. \qquad (*)$$

Makarov's result corresponds to $f(\eta) = |\eta|^2$, which is homogeneous of degree 2. In the future, we want to generalize this result for f which is homogeneous of degree 2 on $\mathbb{C} \setminus \{0\}$ and strictly convex on $\mathbb{C} \setminus \{0\}$ and the measure μ relative to f defined as

$$\int_{\mathbb{C}} \langle \nabla f(\nabla u(z)), \nabla \theta(z) \rangle dA = -\int \theta d\mu. \qquad (*)$$

Plausible Theorem 1

a) μ is concentrated on a set of σ finite H^1 measure.
b) There exists a universal constant $0 < C < \infty$, for any Jordan domain Ω, the measure μ is absolutely continuous with respect to the Hausdorff measure defined relative to λ where

$$\lambda(r) = r \exp\{C\sqrt{\log \frac{1}{r} \log\log\log \frac{1}{r}}\}, \; 0 < t < 10^{-6}$$

The result obtained by Bennewitz and Lewis and Lewis, Nyström, and Poggi-Corradini corresponds to $f(\eta) = |\eta|^p$, which is homogeneous of degree p. We want to obtain similar result for general f, where f is homogeneous of degree p on $\mathbb{C} \setminus \{0\}$ and strictly convex on $\mathbb{C} \setminus \{0\}$;

Plausible Theorem 2

Let $D \subset \mathbb{C}$ be bounded simply connected domain and $1 < p < \infty$, $p \neq 2$. Define

$$\lambda(r) = r \exp\{A\sqrt{\log \frac{1}{r} \log\log \frac{1}{r}}\},\ 0 < r < 10^{-6}$$

Then
a) If $p > 2$, there exists A such that μ defined as in $(*)$ is concentrated on a set of σ finite H^{λ} measure.
b) If $1 < p < 2$, there exists A such that μ defined as in $(*)$ is absolutely continuous with respect to H^{λ}.

Let $v(z) = \log f(\nabla u(z))$. For $k = 1, 2$ we have at z

$$b_{kj} v_{x_j} = \frac{1}{f(\nabla u)} \sum_{n=1}^{2} f_{\eta_n}(\nabla u) b_{kj} u_{x_n x_j}.$$

Summing this over $k, j = 1, 2$, and using $L\zeta = 0$ for $\zeta = u_{x_n}$, we get

$$Lv = \sum_{k,j=1}^{2} \frac{\partial}{\partial x_k}(b_{kj} v_{x_j})$$

and multiplying this by $(f(\nabla u(z)))^2$ we rewrite this equation in the form;

$$(f(\nabla u))^2 Lv = f(\nabla u) T_1 - T_2$$

where at z,

$$T_1 = \sum_{n,j,k,l=1}^{2} b_{nl} b_{kj} u_{x_l x_k} u_{x_j x_n} \quad \text{and} \quad T_2 = \sum_{n,j,k,l=1}^{2} b_{kj} f_{\eta_n} f_{\eta_l} u_{x_l x_k} u_{x_j x_n}$$

We now use matrix notation. We write at z,

$$(b_{kj}(z)) = (f_{\eta_k\eta_j}(\nabla u(z))) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

$$(u_{x_k x_j}(z)) = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} u_{x_1} \\ u_{x_2} \end{pmatrix} = |\nabla u| \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

Observe that if

$$D = \begin{pmatrix} A & B \\ \\ B & C \end{pmatrix} \begin{pmatrix} a & b \\ \\ b & c \end{pmatrix} \text{ then } T_1 = \text{ tr } (D^2). \tag{7}$$

To simplify our calculations we choose an orthonormal matrix O such that

$$O^t \begin{pmatrix} A & B \\ B & C \end{pmatrix} O = \begin{pmatrix} A' & 0 \\ 0 & C' \end{pmatrix}$$
$$O^t \begin{pmatrix} a & b \\ b & c \end{pmatrix} O = \begin{pmatrix} a' & b' \\ b' & c' \end{pmatrix}. \tag{8}$$

Then,

$$T_1 = \text{ tr } D^2 = \text{ tr } [(O^t DO)^2] = (a'A')^2 + 2(b')^2 A'C' + (c'C')^2 \tag{9}$$

$$\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} = O^t \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{10}$$

Then from with $p = 2$, we find at z,

$$f(\nabla u) = \frac{1}{2}|\nabla u|^2[a'(\cos\phi)^2 + 2b'\sin\phi\cos\phi + c'(\sin\phi)^2]$$

Putting everything together we deduce that

$$\begin{aligned} f(\nabla u)T_1 &= \frac{1}{2}|\nabla u|^2[(a'A')^2 + 2(b')^2A'C' + (c'C')^2] \\ &\quad \times[a'(\cos\phi)^2 + 2b'\sin\phi\cos\phi + c'(\sin\phi)^2]. \end{aligned}$$

If we consider T_2, and follow similar procedure we obtain,

$$f(\nabla u)T_1 = T_2.$$

So we have shown that $Lv = 0$ at z when $p = 2$.
The proof of that $Lv \geq 0$ for $p > 2$ and $Lv \leq 0$ for $1 < p < 2$ is essentially same only in these cases we use the fact that f is homogeneous of degree p.

B. Bennewitz and John L. Lewis.
On the dimension of p-harmonic measure.
Ann. Acad. Sci. Fenn. Math,, 30:459–505, 2005.

John L. Lewis, Kaj Nyström, and Pietro Poggi-Corradini.
P harmonic measure in simply connected domains.
Annales de l'institut Fourier, 61:689–715, 2011.

N. Makarov.
Distortion of boundary sets under conformal mapping.
Proc. London Math. Soc., 51:369–384, 1985.