Dimension of p-harmonic measure and related problems

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ODE TO THE P-LAPLACIAN

"I used to be in love with the Laplacian so worked hard to please her with beautiful theorems. However she often scorned me for the likes of Björn Dahlberg, Gene Fabes, Carlos Kenig, and Thomas Wolff. Gradually I became interested in her sister the p Laplacian, $1 < p < \infty$, $p \neq 2$. *I did not find her as pretty as the Laplacian and she was often difficult to handle because of her nonlinearity. However over many years I took a shine to her and eventually developed an understanding of her disposition. Today she is my girl and the Laplacian pales in comparison to her."*

— *John Lewis*

Goals

The size of support of $\rightsquigarrow$ $\left\{ \begin{array}{l} \bullet\ \omega, \text{ harmonic measure} \\ \bullet\ \mu_p, \text{ p-harmonic measure} \\ \bullet\ \mu_f, \text{ Elliptic measure} \end{array} \right\}$ $\rightsquigarrow$ on rough domains in terms of the Hausdorff measure.

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Outline

- Introduction
- Part I: $\sigma-$finiteness of p-harmonic measure in space for $p \geq n$
- Part II: Example of a domain for which $\mathcal{H} - \dim \mu < n - 1$ for $p \geq n$.
- Part III: Related Work

- Let $\Omega \subset \mathbb{R}^n$ be a bounded domain.
- Let N be open neighborhood of $\partial\Omega$.

Fix p, $1 < p < \infty$ and suppose that u is p-harmonic in $\Omega \cap N$. That is, $u \in W^{1,p}(\Omega \cap N)$ and

$$\int \langle |\nabla u|^{p-2} \nabla u, \nabla \phi \rangle \, \mathrm{d}x = 0 \quad \text{for all } \phi \in W_0^{1,p}(\Omega \cap N).$$

If u has continuous second partials in $\Omega \cap N$ and $\nabla u \neq 0$ then u is a classical solution to the p-Laplace equation in $\Omega \cap N$:

$$\Delta_p u := \nabla \cdot \left(|\nabla u|^{p-2} \nabla u \right) = |\nabla u|^{p-4} [(p-2) \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} + |\nabla u|^2 \Delta u] = 0.$$

- This is a degenerate/singular quasilinear elliptic PDE.

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- Assume that $u > 0$ in $\Omega \cap N$.
- Assume $u = 0$ on $\partial\Omega$ in the Sobolev sense.
- Set $u \equiv 0$ in $N \setminus \Omega$. Then u is p-harmonic in N.

By [HKM, Chapter 21] that there exists* a finite, positive, Borel measure μ associated with u whose support contained in $\partial\Omega$ which satisfies

$$\int \langle |\nabla u|^{p-2}\nabla u, \nabla\psi\rangle \,\mathrm{d}x = -\int \psi \,\mathrm{d}\mu \quad \text{for all nonnegative } \psi \in C_0^\infty(N).$$

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Given Borel set E, let

$$L_\delta = \{\text{Covers of } E \text{ with } B(z_i, r_i) \text{ such that } 0 < r_i < \delta\}.$$

- $\mathcal{H}^\lambda_\delta(E)$ denotes the $(\lambda, \delta)-$Hausdorff content of E

$$\mathcal{H}^\lambda_\delta(E) := \inf_{L_\delta} \sum \lambda(r_i)$$

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- Define the Hausdorff dimension of a Borel measure ν by

$$\mathcal{H}-\dim \nu := \inf\{\alpha \,|\, \exists \text{ a Borel set } E \subset \partial\Omega;\ \mathcal{H}^\alpha(E) = 0,\ \nu(\mathbb{R}^n \setminus E) = 0\}$$

i.e., *it is the "smallest dimension" of a set with full ν measure.*

- When everything is smooth, $\mathrm{d}\mu = |\nabla u|^{p-1}\, \mathrm{d}\mathcal{H}^{n-1}|_{\partial\Omega}$.

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Results of interest for harmonic measure

When $p = 2$ and u is the Green's function with pole at $z \in \Omega$ then $\mu = \omega(z, \cdot)$ is harmonic measure with respect to $z \in \Omega$.

- **Carleson**: $\mathcal{H} - \dim \omega = 1$ when $\partial\Omega$ is snowflake in the plane. $\mathcal{H} - \dim \omega \leq 1$ when $\partial\Omega$ is a self similar Cantor set.
- **Jones-Wolff**: $\mathcal{H} - \dim \omega \leq 1$.
- **Wolff**: ω is concentrated on a set of $\sigma-$finite $\mathcal{H}^1$ measure.

Indeed, ω lives on

$$F = \left\{ z \in \partial\Omega : \limsup_{r \to 0} \frac{\omega(B(z, r))}{r} > 0 \right\}.$$

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- $\mathcal{H} - \dim \omega \geq n - 2$ by an easy computation for any domain $\Omega \subset \mathbb{R}^n$.
- **Bourgain**: $\mathcal{H} - \dim \omega \leq n - \tau(n)$ whenever $\Omega \subset \mathbb{R}^n$.
- **Wolff**: $\exists$ *Wolff snowflakes* in $\mathbb{R}^3$ $\begin{cases} \rightsquigarrow \mathcal{H} - \dim \omega > 2, \\ \rightsquigarrow \mathcal{H} - \dim \omega < 2. \end{cases}$
- **Lewis-Verchota-Vogel**: Wolff's result holds in $\mathbb{R}^n$; Harmonic measure on both sides of a Wolff snowflake, say ω_+, ω_- could have

$$\max(\mathcal{H} - \dim \omega_+, \mathcal{H} - \dim \omega_-) < n - 1,$$

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$$\max(\mathcal{H}-\dim\omega_+, \mathcal{H}-\dim\omega_-) < n-1,$$

or

$$\min(\mathcal{H}-\dim\omega_+, \mathcal{H}-\dim\omega_-) > n-1.$$

For general $p \neq 2$;

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Strict inequality holds for $\mathcal{H}-$ dim μ when $\partial\Omega$ is the Von Koch snowflake.

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Results of interest for p-harmonic measure

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To state our recent work we need a notion of n capacity. If $K \subset \overline{B}(x,r)$ is a compact set, define $n-$capacity of K as

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where the infimum is taken over all infinitely differentiable ψ with compact support in $B(x,2r)$ and $\psi \equiv 1$ on K.

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- Let $O \subset \mathbb{R}^n$ be an open set and $\hat{z} \in \partial O$, $\rho > 0$.

- Let $u > 0$ be p-harmonic in $O \cap B(\hat{z}, \rho)$ with continuous zero boundary values on $\partial O \cap B(\hat{z}, \rho)$.

- Extend u to all $B(\hat{z}, \rho)$ by defining $u \equiv 0$ on $B(\hat{z}, \rho) \setminus O$. Then u is p-harmonic in $B(\hat{z}, \rho)$.

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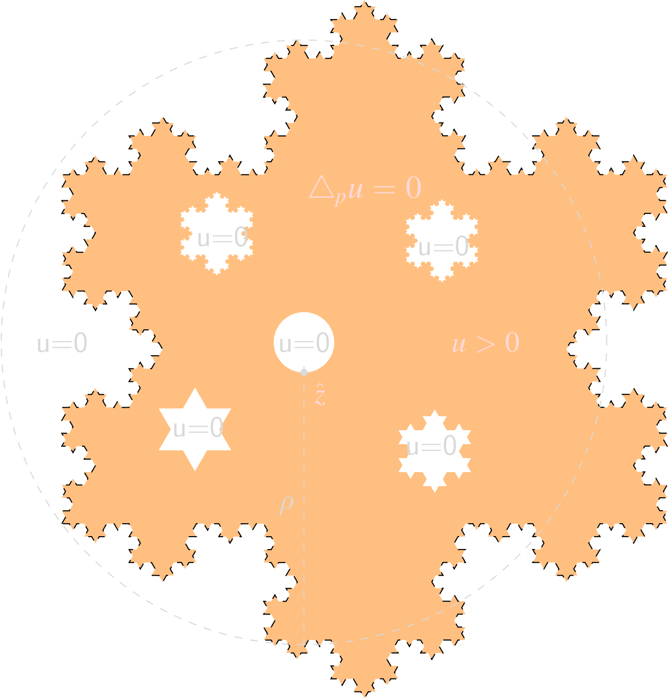
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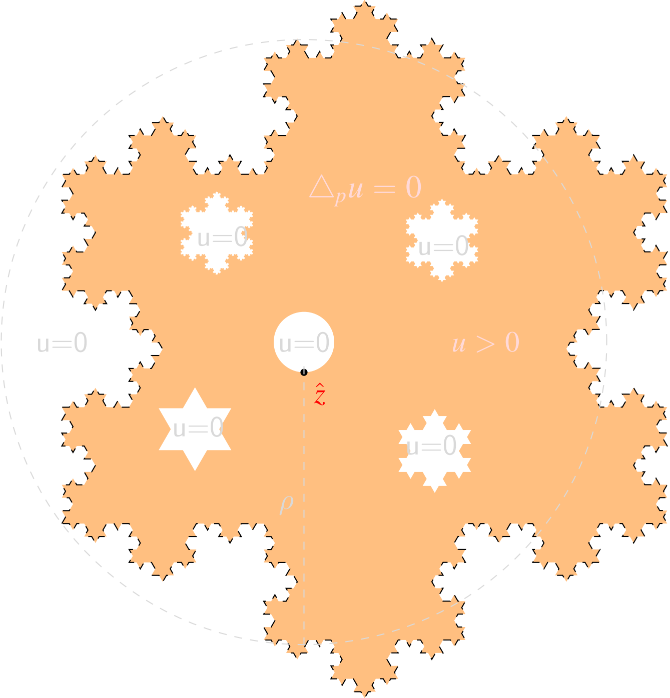

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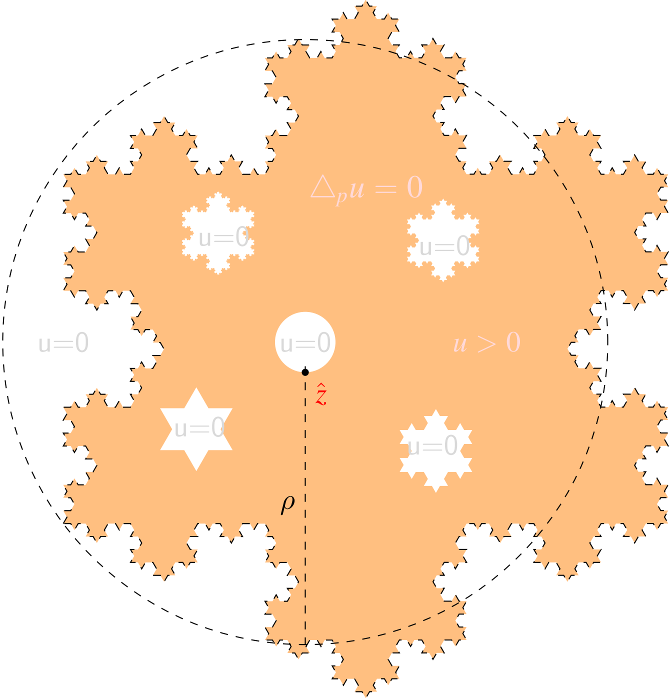

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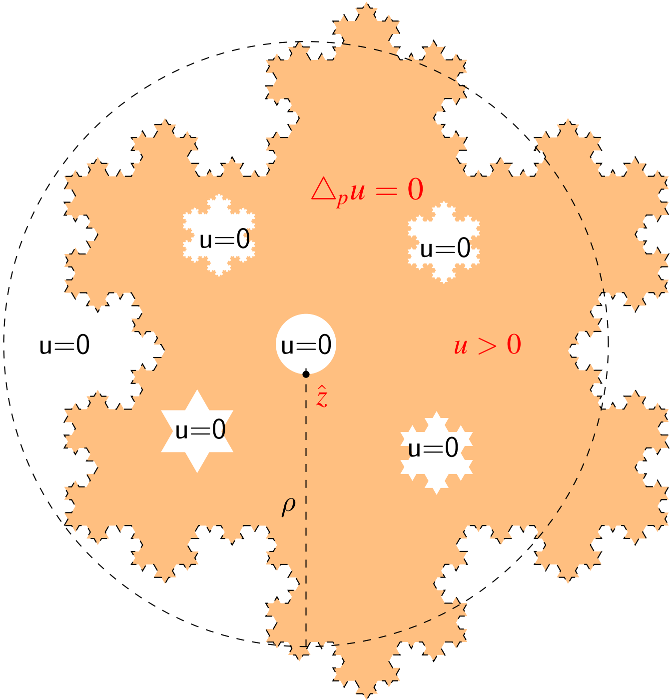

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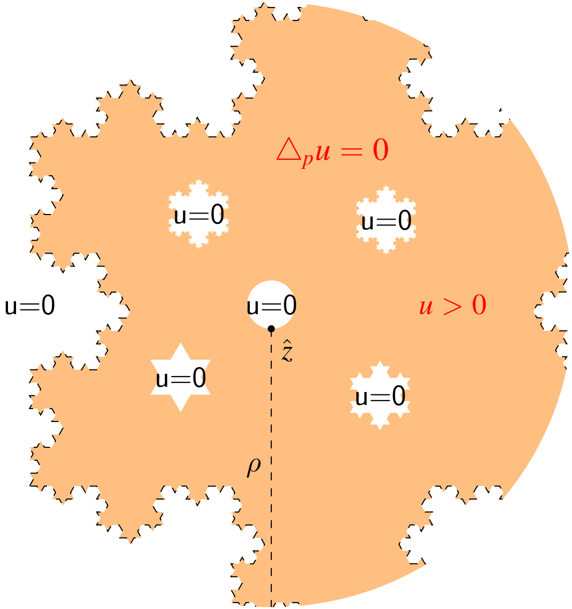
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If $p > n$ then μ is concentrated on a set of $\sigma-$finite $\mathcal{H}^{n-1}$ measure.

Same result holds when $p = n$ provided that $\partial O \cap B(\hat{z}, \rho)$ is locally uniformly fat.

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- If $w \in \partial O$ and $B(w, 4r) \subset B(\hat{z}, \rho)$ then there exists $c = c(p, n) \geq 1$ with

$$\frac{1}{c} r^{p-n} \mu(B(w, r/2)) \leq \max_{B(w,r)} u^{p-1} \leq c r^{p-n} \mu(B(w, 2r)).$$

- The left-hand side is true for any open set O and $p \geq n$.

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Theorem A holds without uniform fatness assumption when $p = n$.

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Sketch of the Proof of Theorem A

Our result follows from this proposition.

Proposition 1

Let λ be a non decreasing function on $[0,1]$ with

$$\lim_{t\to 0}\frac{\lambda(t)}{t^{n-1}}=0.$$

There exists $c=c(p,n)$ and a set $Q\subset\partial O\cap B(\hat{z},\rho)$ such that

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We first show how our result follows from this proposition.

- $\mathcal{H}^{n-1}(\mathcal{P}_m) < \infty$ for each positive integer m where

$$\mathcal{P}_m := \left\{ x \in \partial O \cap B(\hat{z}, \rho) : \ \limsup_{t \to 0} \frac{\mu(B(x,t))}{t^{n-1}} > \frac{1}{m} \right\}.$$

Therefore,

$$\mathcal{P} = \left\{ x \in \partial O \cap B(\hat{z}, \rho) : \ \limsup_{t \to 0} \frac{\mu(B(x,t))}{t^{n-1}} > 0 \right\}$$

$\sigma-$finite $\mathcal{H}^{n-1}$ measure.

- Need to show: $\mu(Q \setminus P) = 0$.

- From Proposition 1 and measure theoretic arguments there exists a Borel set $Q_1 \subset Q$ with

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Sketch of the Proof of Proposition 1

Proposition 1

Let λ be a non decreasing function on $[0,1]$ with

$$\lim_{t \to 0} \frac{\lambda(t)}{t^{n-1}} = 0.$$

There exists $c = c(p,n)$ and a set $Q \subset \partial O \cap B(\hat{z}, \rho)$ such that

$$\mu(\partial O \cap B(\hat{z}, \rho) \setminus Q) = 0$$

and for every $w \in Q$ there exists arbitrarily small $r = r(w) > 0$ and a compact set $F = F(w,r)$such that

$$\mathcal{H}^{\lambda}(F) = 0 \text{ and } \frac{1}{c} \leq \mu(F).$$

Sketch of the Proof of Proposition 1

- Translation, dilation invariance of the p-Laplacian and a measure theoretic argument to reduce the proof of Proposition to the situation when $w = 0$, $B(0, 100) \subset B(\hat{z}, \rho)$.

- There is some c = c(p, n) and $2 \leq t \leq 50$ such that

$$\frac{1}{c} \leq \mu(B(0,1)) \leq \max_{B(0,2)} u \leq \max_{B(0,t)} u \leq c\mu(B(0,100)) \leq c^2.$$

To finish the proof of Proposition 1, it suffices to show for given small $\epsilon, \tau > 0$ that there exists a Borel set $E \subset \partial O \cap B(0,20)$ and $c = c(p,n) \geq 1$ with

$$\mathcal{H}^{\lambda}_{\tau}(E) \leq \epsilon \text{ and } \mu(E) \geq \frac{1}{c}.$$

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- Let M a large positive number and $s < e^{-M}$.

For each $z \in \partial O \cap B(0,15)$ there is $t = t(z)$, $0 < t < 1$ with either

$$(\alpha)\ \mu(B(z,t)) = Mt^{n-1}, t > s$$

or

$$(\beta)\ t = s.$$

- Use the Besicovitch covering theorem to get a covering $B(z_j, t_j)_1^N$ of $\partial O \cap B(0,15)$ where $t_j = t(z_j)$ is the maximal for which either (α) or (β) holds.

A stopping time argument

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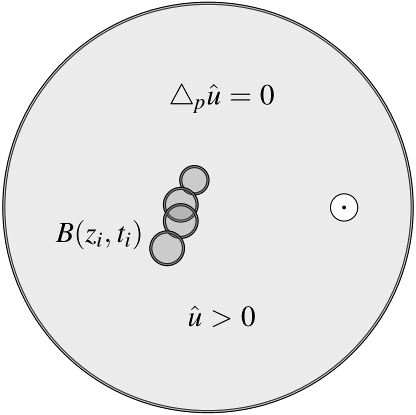
$$\Omega := O \cap B(0,15) \setminus \bigcup_{i=1}^{N} \overline{B}(z_i, t_i) \text{ and } D := \Omega \setminus \overline{B}(\tilde{z}, 2r_1)$$



- Let $\hat{u}$ be the p-harmonic function in D with continuous boundary values, $\hat{u} = \min_{\overline{B}(\tilde{z}, 2r_1)} u$ on $\partial\overline{B}(\tilde{z}, 2r_1)$ and $\hat{u} = 0$ on $\partial\Omega$.
- Let $\hat{\mu}$ be the p-harmonic measure associated with $\hat{u}$.

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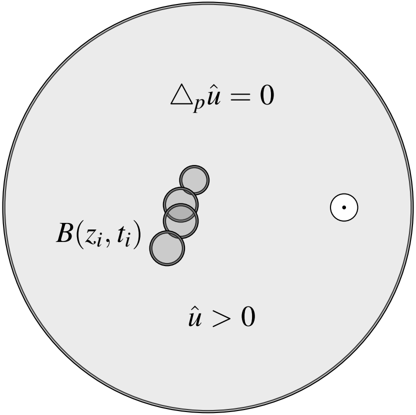

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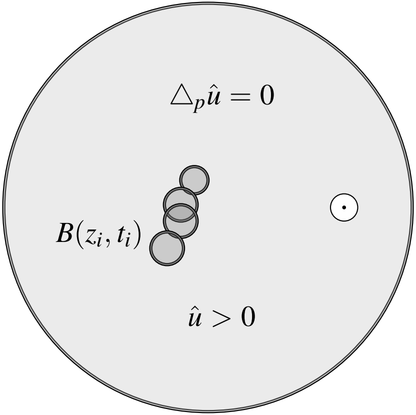

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- $\hat{u} \le u$ in D.

- $\partial\Omega$ is smooth except for a set of finite $\mathcal{H}^{n-2}$.

- Using some barrier type estimate one can also show

$$|\nabla \hat{u}| \le cM^{\frac{1}{p-1}} \text{ in } D.$$

- Combining these we can show

$$t_j^{1-n}\hat{\mu}(\overline{B}(z_j,t_j)) \le c t_j^{1-p} \max_{B(z_j,2t_j)} u^{p-1} \le c^2 t_j^{1-n}\mu(B(z_j,4t_j)).$$

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- Let $A >> 1$ be given

- $\{1, \ldots, N\}$ can be divided into disjoint subsets $\mathcal{G}, \mathcal{B}$, and $\mathcal{U}$ as

$$\begin{cases} \mathcal{G} := \{j : \, t_j > s\}, \\ \\ \mathcal{B} := \{j : \, t_j = s \text{ and } |\nabla \hat{u}|^{p-1} \geq M^{-A} \text{ for some } x \in \partial\Omega \cap \partial B(z_j, t_j)\}, \\ \\ \mathcal{U} := \{j : \, j \text{ is not in } \mathcal{G} \text{ or } \mathcal{B}\}. \end{cases}$$

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$$\int_{\partial\Omega} |\nabla \hat{u}|^{p-1} \, |\log |\nabla \hat{u}|| \, \mathrm{d}\mathcal{H}^{n-1} \leq c' \log M.$$

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Part II: Example of a domain for which $\mathcal{H}-\dim \mu < n-1$ for $p \geq n$.

There is an unpublished result of Jones-Wolff in [GM, Chapter IX, Theorem 3.1];

- **Jones-Wolff**: Let $\Omega = \mathbb{C} \cup \{\infty\} \setminus \mathcal{C}$ where $\mathcal{C}$ is a "*Cantor like*" compact set. Then $\mathcal{H}-\dim \omega < 1$.

Our aim is to generalize this result to p-harmonic measure, μ, in $\mathbb{R}^n$ for $p \geq n \geq 2$ and for a certain domain.

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Construction of the domain

Let S' be the square with side length $1/2$ and center 0 in $\mathbb{R}^n$. $\mathcal{C}_0 := S'$.

Let $Q_{11}, \ldots, Q_{14}$ be the squares of the four corners of $\mathcal{C}_0$ of side length a_1, $0 < \alpha < a_1 < \beta < 1/4$, and let $\mathcal{C}_1 = \bigcup_{i=1}^{4} Q_{1i}$.

Let $\{Q_{2j}\}$, $j = 1, \ldots, 16$ be the square of corners of each Q_{1i}, $i = 1, \ldots, 4$ of side length $a_1 a_2$, $\alpha < a_2 < \beta$. Let $\mathcal{C}_2 = \bigcup_{j=1}^{16} Q_{2j}$.

By repeating the process we get $\mathcal{C}$.

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By repeating the process we get $\mathcal{C}$.

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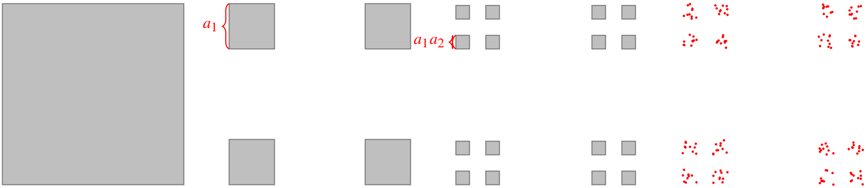

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New Result

Let $\mathcal{S} = 2S' \subset \mathbb{R}^n$ and let u^∞ be a p-harmonic function in $\mathcal{S} \setminus \mathcal{C}$ with boundary values $u^\infty = 1$ on $\partial\mathcal{S}$ and $u^\infty = 0$ on $\mathcal{C}$. Let μ^∞ be the p-harmonic measure associated to u^∞.

Theorem B (A.-Lewis-Vogel)

Then $\mathcal{H} - \dim \mu^\infty \leq n - 1 - \delta$ for some $\delta = \delta(p, n, \alpha, \beta) > 0$.

- Here $\delta \geq c^{-1}(p - n)$ where $c \geq 1$ can be chosen to depend only on n, α, and β when $p \in [n, n+1]$.

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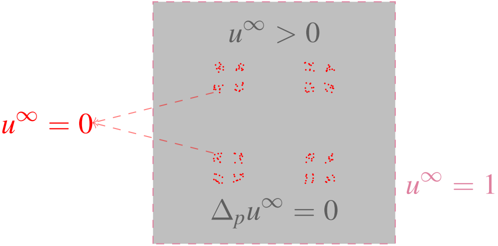

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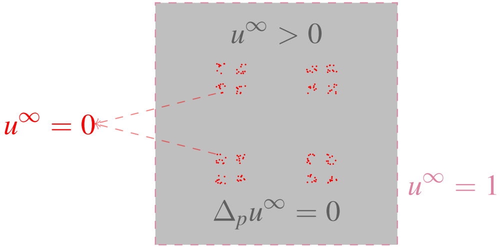

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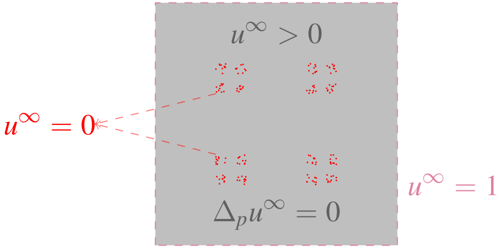

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Sketch of Proof of Theorem B

- Jones and Wolff used the idea of counting zeros of ∇G.
- In higher dimensions and when $p \neq 2$ we then have a little control over the zeros of ∇u.

Let

$$\tilde{\Gamma} = \left\{ \tilde{Q}_{k,j};\ k = 1, \ldots,\ \text{and}\ j = 1, \ldots, 2^{kn} \right\}.$$

Our result essentially follows from this Proposition;

Proposition 2

Let $\tilde{Q} \in \tilde{\Gamma}$ be a given cube. Then there exists $\delta' > 0$ with the same dependence as δ in Theorem B, $c = c(p, n, \alpha, \beta) \geq 1$, and a compact set $F \subset \mathcal{C} \cap \tilde{Q}$ with

$$\mathcal{H}^{n-1-\delta'}(F) = 0 \text{ and } \frac{1}{c} \leq \mu^{\infty}(F).$$

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In [HKM, Chapter 21], it was shown that the measure associated with a positive weak solution u with 0 boundary values for a larger class of qusailinear elliptic PDEs exists;

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where $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies certain structural assumptions.

The measure is so called $\mathcal{A}$-harmonic measure.

In [BL, Closing remarks 10], the authors pointed out this fact and asked for what PDE one can obtain dimension estimates on the associated measure.

If $\mathcal{A}(\xi) = |\xi|^{p-2}\xi$, then the above PDE becomes the usual p-Laplace equation.

$$\{\text{Laplace}\} \subsetneqq \{\text{p-Laplace}\} \subsetneqq \{\triangle_f u = 0\} \subsetneqq \{\mathcal{A} - \text{Harmonic PDEs}\}.$$

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Introduction to $\Delta_f u = 0$

Let p be fixed, $1 < p < \infty$. Let f be a function with following properties;

1. $f : \mathbb{R}^n \to (0, \infty)$ is homogeneous of degree p.

$$\text{That is, } f(\eta) = |\eta|^p f\left(\frac{\eta}{|\eta|}\right) > 0 \text{ when } \eta \in \mathbb{R}^n \setminus \{0\}.$$

2. $\mathcal{D}f = (f_{\eta_1}, \ldots, f_{\eta_n})$ has continuous partial derivatives when $\eta \neq 0$.
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We consider weak solutions, u, to the Euler Lagrange equation;

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in $\Omega \cap N$ where N is an open neighborhood of $\partial\Omega$. Assume also that $u > 0$ in $N \cap \Omega$ with continuous boundary values on $\partial\Omega$. Set $u \equiv 0$ in $N \setminus \Omega$ to have $u \in W^{1,p}(N)$ and $\triangle_f u = 0$ weakly in N. Then, there exists a unique finite positive Borel measure μ_f associated with u having support contained in $\partial\Omega$ satisfying

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$$\mathcal{L}\zeta = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(f_{\eta_i \eta_j} \zeta_j \right)$$

Then

- $\zeta = u$ is a weak solution to $\mathcal{L}\zeta = 0$
- $\zeta = u_{x_k}$ for $k = 1, \ldots, n$ is weak solution to $\mathcal{L}\zeta = 0$
- $\zeta = \log f(\nabla u)$ is a weak sub solution and weak solution to $\mathcal{L}\zeta = 0$

respectively when $p > n$ and $p = n$.

Using this sub solution estimate and following arguments we have used for p harmonic measure we show that

Theorem C (A.-Lewis-Vogel)

Theorem A and Theorem B hold for μ_f.

More results in the plane for simply connected domains

Let $n = 2$ and $\Omega \subset \mathbb{R}^2$ be any bounded simply connected domain.

Let

$$\lambda(r) := r \exp\left\{ A \sqrt{\log \frac{1}{r} \log\log\log \frac{1}{r}} \right\}.$$

- **Makarov**: $\omega \ll \mathcal{H}^{\lambda}$ if A is large enough.

For any small $\epsilon > 0$,

$$\omega \ll \mathcal{H}^{1-\epsilon} \text{ and } \omega \perp \mathcal{H}^{1+\epsilon}.$$

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- **Lewis-Nyström-Poggi Corradini**:

1. $\mu_p \ll \mathcal{H}^{\hat{\lambda}}$ when $1 < p < 2$ for some $A = A(p) \geq 1$.

2. μ_p is concentrated on a set of σ−finite $\mathcal{H}^{\hat{\lambda}}$ when $2 < p < \infty$ for some $A = A(p) \leq -1$.

- **Lewis**:

1. If $1 < p < 2$, then $\mu_p \ll \mathcal{H}^{\lambda}$ for $A = A(p)$ sufficiently large.

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$$\mathcal{H} - \text{dim}\ \mu_p \begin{cases} \geq 1 & \text{when } 1 < p < 2, \\ = 1 & \text{when } p = 2, \\ \leq 1 & \text{when } 2 < p < \infty. \end{cases}$$

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- let $u > 0$ be a weak solution to $\triangle_f u = 0$ in $\Omega \cap N$. Let μ_f be the measure associated with u.
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1. $\hat{u}_z$ is a $K-$quasiregular mapping.
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Capacitary function

$\hat{u}$ is called capacitary function for D if $\hat{u}$ is positive weak solution to $\triangle_f \hat{u} = 0$ in D with continuous boundary values $\hat{u} = 1$ on $\partial\overline{B(z_0, d(z_0, \partial\Omega)/2)}$ and $\hat{u} = 0$ on $\partial\Omega$.

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Quasiregular mapping

$\hat{u}_z \in W^{1,2}$ locally and $|\hat{u}_{\bar{z}}| \leq k|\hat{u}_z|$ a.e. in Ω, and $k = (K-1)/(K+1)$ where

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x_1} - \mathrm{i}\frac{\partial}{\partial x_2}\right) \text{ and } \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x_1} + \mathrm{i}\frac{\partial}{\partial x_2}\right)$$

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Endpoint-type results for $\mathcal{H}-\dim \mu_f$

A weaker version of Makarov's and Lewis' results;

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Some Questions

Let $\mathcal{C}$ be the *four-corner Cantor set* with $\{a_i\}$ where each a_i satisfies $\alpha < a_i < \beta \leq 1/2$ in $\mathbb{R}^n$.

Let $\Omega = \mathbb{C} \cup \{\infty\} \setminus \mathcal{C}$.

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Let $\mathcal{S}$ be the unit cube centered at 0 in $\mathbb{R}^n$ and let $\mathcal{C}$ be the four-corner Cantor set. Let u be p-harmonic in $\mathcal{S} \setminus \mathcal{C}$ with continuous boundary values 1 on $\partial\mathcal{S}$ and 0 on $\mathcal{C}$. Let μ be the p-harmonic measure associated with u.

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$\alpha \to 1/2 \implies \mathcal{H} - \dim \mu_p \to n-1$ in $\mathbb{R}^n$?

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Conjecture (Øksendal)

$\mathcal{H} - \dim \omega < n - 1$ for any domain $\Omega \subset \mathbb{R}^n$.

This conjecture is false due to a result of Wolff; $\mathcal{H} - \dim \omega > n - 1$ for some snowflake domains.

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What is the best value of τ?

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$$\mathcal{H} - \dim \omega \leq n - 1 + \frac{n-2}{n-1}.$$

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