A Minkowski problem for nonlinear capacity

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April 22, Boston AMS special session

Intro 1, this is joint work

Title of Paper on ArXiv:
The Brunn-Minkowski inequality and a Minkowski problem for nonlinear capacity.
with Murat Akman, Jasun Gong, Jay Hineman, John Lewis

Abstract of Talk: We focus on the Minkowski problem in $\mathbb{R}^n$ for classes of equations similar to and including the p-Laplace equations for $1 < p < n$. The minimization problem that leads to the solution will be described along with a discussion of why the minimizing set has nonempty interior for the full range $1 < p < n$. We may briefly discuss the Brunn-Minkowski inequality which leads to uniqueness arguments for the Minkowski problem, and is helpful in deriving the Hadamard Variational formula.

Intro 2, credits

Much of this talk is inspired by Jerison's paper
A Minkowski problem for electrostatic capacity in Acta Math.
This is the $p = 2$ case.
and by
The Hadamard variational formula and the Minkowski problem for p-capacity by Colesanti, Nyström, Salani, Xiao, Yang, Zhang in Advances in Mathematics
This is the $1 < p < 2$ case.

The Brunn-Minkowski part is inspired by Colesanti, Salani
The Brunn-Minkowski inequality for p-capacity of convex bodies. in Math. Ann.

See Jasun Gong's talk for that! Special Session on Analysis and Geometry in Non-smooth Spaces, IV at 3:00pm

Intro 3, credits

Lewis and Nyström have several papers concerning the boundary behavior of p-harmonic functions, some of those results needed extensions to this setting. In addition they have recent work on the behavior on lower dimensional sets $k < n - 1$, which we also need.

Regularity and free boundary regularity for the p-Laplace operator in Reifenberg flat and Ahlfors regular domains. J. Amer. Math. Soc.

Quasi-linear PDEs and low-dimensional sets. to appear JEMS

Venouziou and Verchota, have a result that we extend and use to get nonempty interiors in the $k = n - 1$ dimensional case. *The mixed problem for harmonic functions in polyhedra of* $\mathbb{R}^3$.

For even more, see John Lewis's talk, here, next!!

Nonlinear Capacity $1 < p < n$

We are thinking of $\mathbb{R}^n$ with $1 < p < n$ and a p homogeneous function

$$f(t\eta) = t^p f(\eta) \text{ for all } \eta \in \mathbb{R}^n \setminus \{0\} \text{ and } t > 0$$

For example, the p-Laplacian comes from,

$$f(\eta) = \frac{1}{p}|\eta|^p \text{ so } Df(\eta) = |\eta|^{p-2}\eta$$

and for a function $u(x)$, $x \in \mathbb{R}^n$

$$\mathrm{div}(Df(\nabla u)) = \nabla \cdot |\nabla u|^{p-2}\nabla u$$

More generally f could be convex but **not** rotationally invariant

$$f(\eta) = (1 + \frac{\epsilon\eta_1}{|\eta|})|\eta|^p$$

Nonlinear capacity, conditions on $\mathcal{A} = Df$

In general we have $\mathcal{A}(\eta) = Df(\eta)$ mapping $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ with continuous first partials satisfying for some $1 < p < n$ and some $\alpha \geq 1$

$$\alpha^{-1}|\eta|^{p-2}||\xi|^2 \leq \sum_{i,j=1}^{n} \frac{\partial \mathcal{A}_i(\eta)}{\partial \eta_j}\xi_i\xi_j \leq \alpha|\eta|^{p-2}|\xi|^2$$

and

$$\mathcal{A}(\eta) = |\eta|^{p-1}\mathcal{A}(\eta/|\eta|)$$

For uniqueness in BM and so uniqueness in M we need

$$|\frac{\partial \mathcal{A}_i(\eta)}{\partial \eta_j} - \frac{\partial \mathcal{A}_i(\eta')}{\partial \eta_j}| \leq \Lambda|\eta - \eta'||\eta|^{p-3}$$

For some $\Lambda \geq 1$, $1 \leq i, j \leq n$, $0 < \frac{1}{2}|\eta| \leq |\eta'| \leq 2|\eta|$.

Nonlinear capacity see Heinonen Kilpeläinen Martio

Nonlinear Potential Theory of Degenerate Elliptic Equations

For E a convex, compact subset of $\mathbb{R}^n$, let $\Omega = E^c$ then

$$\mathrm{Cap}_{\mathcal{A}}(E) = \inf_{\substack{\psi \in C_0^\infty \\ \psi|_E \geq 1}} \int_{\mathbb{R}^n} f(\nabla \psi) dx$$

For $f(\eta) = \frac{1}{p}|\eta|^p$ this is the p-capacity, Cap_p. From our assumptions on $\mathcal{A}$

$$Cap_p(E) \approx Cap_{\mathcal{A}}(E)$$

where the constant of equivalence depends only on p, n, α.
For $\mathrm{Cap}_{\mathcal{A}}(E) > 0$ (equivalently $\mathcal{H}^{n-p}(E) = \infty$) there is a unique continuous u attaining the inf, $0 < u \leq 1$ on $\mathbb{R}^n$, u is $\mathcal{A}$-harmonic in Ω, $u = 1$ on E,..., u is the $\mathcal{A}$-capacitary function of E.

Nonlinear capacity, tricks!

For the $\mathcal{A}$-capacitary function u of E it's important to consider the function $1-u$, this function is positive in Ω and 0 on $\partial\Omega$ but it is not in general an $\mathcal{A}$-harmonic function. Luckily, it is $\tilde{\mathcal{A}}(\eta) = -\mathcal{A}(-\eta)$-harmonic, and $\tilde{\mathcal{A}}$ satisfies the same condtions as $\mathcal{A}$ with the same constants.

If $\hat{E} = \rho E + z$, a scaled and translated E, then
$\hat{u}(x) = u((x-z)/\rho)$ is the $\mathcal{A}$-capacitary function of $\hat{E}$ and
$\mathrm{Cap}_{\mathcal{A}}(\hat{E}) = \rho^{n-p}\mathrm{Cap}_{\mathcal{A}}(E)$
What about rotations? See the trick above!

For E convex, compact, subset of $\mathbb{R}^n$ the dimension of E (at every point of E) is some integer k, then $H^k(E) < \infty$.
• for $\mathrm{Cap}_{\mathcal{A}}(E) > 0$ we need $H^{n-p}(E) = \infty$ and therefore $n-p < k$, or $n-k < p < n$.

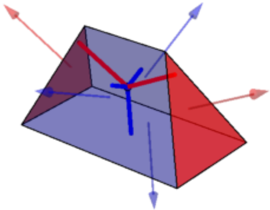
Hadamard variational formula

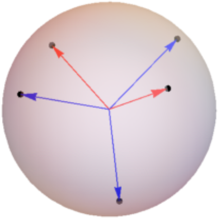
For convex compact sets E_1, E_2 with $0 \in E_1$, (not necessarily $0 \in E_1^\circ$) and $0 \in E_2^\circ$, and $t \geq 0$ we have

$$\frac{d}{dt}\mathrm{Cap}_{\mathcal{A}}(E_1 + tE_2)\Big|_{t=t_2} = (p-1)\int_{\partial(E_1+t_2E_2)} h_2(g(x))f(\nabla u(x))dH^{n-1}$$

h_2 is the support function of E_2, g is the Gauss map of $E_1 + t_2E_2$ and u is the $\mathcal{A}$-capacitary function of $E_1 + t_2E_2$. Here we are varying off the base configuration $E_1 + t_2E_2$ by $(t - t_2)E_2$.
And we use the Brunn-Minkowski inequality in this proof! It says that $\mathrm{Cap}_{\mathcal{A}}^{1/(n-p)}(E_1 + tE_2)$ is concave in t.

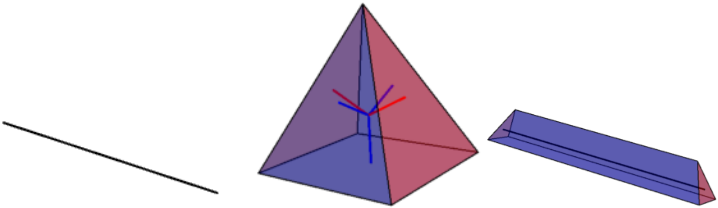
Polyhedron, Gauss map, support function.





Gauss map: 2 red faces (right, left) and 3 blue faces (front, bottom $= F_1$, back) for $x \in F_1$, $g(x) = -e_3$, $g^{-1}(-e_3) = F_1$.
Support function: for $x \in$ bottom face, $h(g(x))$ is the distance of the face to the origin, the length of the vertical thick blue segment.
Next Slide: Move the 3 blue faces to the origin, the solid blue segments shrink to zero, call this E_1. Make all the solid segments the same length, call this E_2.

Polyhedron example E_1, E_2 and $E_1 + t_2 E_2$



- E_2 has five unit normals $\xi_1, \ldots, \xi_5$ all with $h_2(\xi_k) = a$

On the faces F_i, $i = 1, \ldots, 5$ of $E_1 + t_2 E_2$ the integral above is

$$(p-1) \sum_{i=1}^{5} a \int_{F_i} f(\nabla u(x)) dH^{n-1}$$

u is the $\mathcal{A}$-capacitary function.

Does $f(\nabla u(x))$ make sense in the boundary integral?

Use the $1-u$ trick above, this is positive, 0 on the boundary has an associated measure...
In the harmonic case, $p=2$, $\int_{\partial\Omega} |\nabla u| dH^{n-1}$ gives a "harmonic measure at infinity" = Capacity of E and by results of Dahlberg

$$\int_{\partial\Omega} |\nabla u|^2 dH^{n-1} \leq c \left(\int_{\partial\Omega} |\nabla u| dH^{n-1} \right)^2$$

in the p-harmonic setting this becomes

$$\int_{\partial\Omega} |\nabla u|^p dH^{n-1} \leq c \left(\int_{\partial\Omega} |\nabla u|^{p-1} dH^{n-1} \right)^{\frac{p}{p-1}}$$

where the constant depends on the Lipschitz nature, meaning the Lipschitz constant **and** the number of balls used.
• As n-d polyhedron shrink to $(k < n)$-d polyhedron keeping the Lipschitz constant fixed, the number of balls $\to \infty$ and c blows up.

Hadamard- capacity formula

In case $E_1 = E_2 = E_0$ and $t = 0$ this says

$$\frac{d}{dt}\mathrm{Cap}_{\mathcal{A}}(E_0 + tE_0)\bigg|_{t=0} = (p-1)\int_{\partial E_0} h(g(x))f(\nabla u(x))dH^{n-1}$$

Where h, g and u are the support, Gauss, and capacitary functions for E_0.
But the LHS is just

$$\frac{d}{dt}\bigg|_{t=0}(1+t)^{n-p}\mathrm{Cap}_{\mathcal{A}}(E_0) = (n-p)\mathrm{Cap}_{\mathcal{A}}(E_0)$$

so

$$\mathrm{Cap}_{\mathcal{A}}(E_0) = \frac{p-1}{n-p}\int_{\partial E_0} h(g(x))f(\nabla u(x))dH^{n-1}$$

For a polyhedron

For E_0 a polyhedron with $0 \in E_0^\circ$, with m faces $F_1, \ldots, F_m$ with unit outer normals $\xi_1, \ldots, \xi_m$ this gives

$$\mathrm{Cap}_{\mathcal{A}}(E_0) = \frac{p-1}{n-p} \sum_{i=1}^{m} \int_{F_i} h(\xi_i) f(\nabla u) dH^{n-1}$$

Now $h(\xi_i)$ is the distance of support plane with normal ξ_i to the origin, that means for $x \in F_i$, $h(\xi_i) = x \cdot \xi_i = q_i$

$$\mathrm{Cap}_{\mathcal{A}}(E_0) = \frac{p-1}{n-p} \sum_{i=1}^{m} q_i \int_{F_i} f(\nabla u) dH^{n-1}$$

set $c_i = \int_{F_i} f(\nabla u) dH^{n-1}$ we have

$$\mathrm{Cap}_{\mathcal{A}}(E_0) = \frac{p-1}{n-p} \sum_{i=1}^{m} q_i c_i$$

Capacity is Translation invariant

Translating E_0 by x, then $\mathrm{Cap}_{\mathcal{A}}(E_0 + x) = \mathrm{Cap}_{\mathcal{A}}(E_0)$ but the support function of $E_0 + x$ is $h(\xi) + x \cdot \xi$ so that

$$\frac{p-1}{n-p} \sum_{i=1}^{m} q_i c_i = \frac{p-1}{n-p} \sum_{i=1}^{m} (q_i + x \cdot \xi_i) c_i$$

which gives, for all x,

$$\sum_{i=1}^{m} (x \cdot \xi_i) c_i = 0$$

and therefore

$$\sum_{i=1}^{m} \xi_i c_i = 0$$

The Minkowski problem- discrete case

The setup: Let μ be a finite positive Borel measure on the unit sphere $\mathbb{S}^{n-1}$ given by

$$\mu(K) = \sum_{i=1}^{m} c_i \delta_{\xi_i}(K) \text{ for all Borel } K \subset \mathbb{S}^{n-1}$$

where the $c_i > 0$, the ξ_i are distinct unit vectors, δ_{ξ_i} is a unit mass at ξ_i.

The Question: Is there a compact, convex, set E_0 with nonempty interior so that

$$\mu(K) = \int_{g^{-1}(K)} f(\nabla u) dH^{n-1}$$

where g and u are the Gauss and capacitary functions for E_0?

Jerison $p = 2$

Let $n \geq 3$ and $f(\eta) = \frac{1}{2}|\eta|^2$, this gives the Laplacian, and so harmonic functions u, and the usual electrostatic capacity of E.

If μ satisfies (i) $\sum_{i=1}^m c_i|\theta \cdot \xi_i| > 0$ and (ii) $\sum_{i=1}^m c_i\xi_i = 0$ then there is a compact, convex set E with nonempty interior so that

$$\mu(K) = \int_{g^{-1}(K)} |\nabla u|^2 dH^{n-1} \text{ for all Borel } K \subset \mathbb{S}^{n-1}$$

When $n > 4$ the set E is unique up to translation, when $n = 3$ there is a $b > 0$ so that the equation holds with b on the right hand side, and then E is unique up to translation and dilation.

Why (i)?

We've seen why (ii), how about (i)?
This condition is used to show that for $0 \le q_i < \infty$, sets like $E(q) = \bigcap_{i=1}^m \{x \mid x \cdot \xi_i \le q_i\}$ are bounded.
(ii) says $\int_{\mathbb{S}^{n-1}} \theta \cdot \xi d\mu = \theta \cdot \sum_{i=1}^m c_i \xi_i = 0$ for all $\theta \in \mathbb{S}^{n-1}$
so

$$\int_{\mathbb{S}^{n-1}} (\theta \cdot \xi)^+ d\mu = \int_{\mathbb{S}^{n-1}} (\theta \cdot \xi)^- d\mu$$

(i) says $0 < \sum_{i=1}^m c_i |\theta \cdot \xi_i| = \int_{\mathbb{S}^{n-1}} |\theta \cdot \xi| d\mu = 2 \int_{\mathbb{S}^{n-1}} (\theta \cdot \xi)^+ d\mu$
so

$$\int_{\mathbb{S}^{n-1}} (\theta \cdot \xi)^+ d\mu > c_0 > 0$$

For $\tau \in \mathbb{S}^{n-1}$ and $x = r\tau \in E(q)$, $r \ge 0$

$$rc_0 < \int_{\mathbb{S}^{n-1}} (r\tau \cdot \xi)^+ d\mu = \sum_{i=1}^m (r\tau \cdot \xi_i)^+ c_i \le \sum_{i=1}^m q_i c_i = \gamma(q)$$

So that $E(q) \subseteq \overline{B(0, \gamma(q)/c_0)}$

Colesanti, Nyström, Salani, Xiao, Yang, Zhang, $1 < p < 2$

Let $n \geq 3$ and $f(\eta) = \frac{1}{p}|\eta|^p$, this gives the p-Laplacian, and so p-harmonic functions u, and the usual p-capacity of E.

If μ satisfies (i) $\sum_{i=1}^m c_i|\theta \cdot \xi_i| > 0$ and (ii) $\sum_{i=1}^m c_i\xi_i = 0$ and (iii) for all $\xi \in \mathbb{S}^{n-1}$ if $\mu(\{\xi\}) \neq 0$ then $\mu(\{-\xi\}) = 0$ then there is a compact, convex set E with nonempty interior so that

$$\mu(K) = \int_{g^{-1}(K)} |\nabla u|^p dH^{n-1} \text{ for all Borel } K \subset \mathbb{S}^{n-1}$$

E is unique up to translation.

The minimization procedure

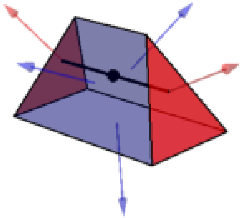
For $q_i \geq 0$ let

$$\begin{aligned} E(q) &= \bigcap_{i=1}^{m} \{x \mid x \cdot \xi_i \leq q_i\} \\ \Theta &= \{E(q) \mid \mathrm{Cap}_{\mathcal{A}}(E(q)) \geq 1\} \\ \gamma(q) &= \sum_{i=1}^{m} q_i c_i \\ \gamma &= \inf_{E(q) \in \Theta} \gamma(q) \end{aligned}$$

Because of condition (i) the $E(q) \in \Theta$ are bounded, compact, convex sets.
There is a sequence $q^k \to \hat{q}$ so that $E(q^k) \to E(\hat{q}) = E_1$ a convex, compact set with $\gamma = \gamma(\hat{q})$
Is E_1° nonempty? Do we have $\hat{q}_i > 0$ for $i = 1, \ldots, m$?

Recall the examples

• Imagine the 3 blue faces moving to the origin and giving the minimizer E_1 as the black 1-d segment. The $\hat{q}_i$ for the blue faces are all 0.



• Or imagine that the two red faces are parallel and that they move to the origin, giving a 2-d set for the minimizer E_1. The $\hat{q}_i$ for the red faces are now 0.

• In either case, for appropriate p, $\mathrm{Cap}_{\mathcal{A}}(E_1) = 1$ is possible!

The minimizer E_1 has nonempty interior, $1 < p \le 2$

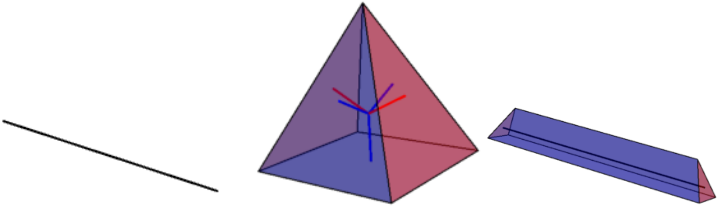
Given condition (iii) , NO ANTIPODAL NORMALS
• If E_1 is $k = n - 1$ dimensional then there must of have been two opposing normals $\xi_i = -\xi_j$, a contradiction.
• If E_1 is $k \le n - 2$ dimensional then $n - p \ge n - 2 \ge k$ so $H^{n-p}(E_1) < \infty$ and E_1 has 0 $\mathcal{A}$-capacity, a contradiction.

Jerison for $p = 2$ uses condition (iii), but it is not necessary as an inradius estimate can be used to get nonempty interior.

Colesanti et al need (iii) in the $k = n - 1$ case when $p \neq 2$. And they need $1 < p \le 2$ for the $k < n - 1$ situation.

The minimizer E_1 has nonempty interior, $1 < p < n$

For $k < n-1$, a situation illustrated here



We set $E_2 = \bigcap_{i=1}^{m} \{x \mid x \cdot \xi_i \leq a\}$ and consider $E_1 + tE_2$
It turns out that for $q_i(t) = (\hat{q}_i + at)/\mathrm{Cap}_{\mathcal{A}}(\tilde{E}(t))$
$\gamma(q(t)) \leq k(t) < \gamma$ for $t > 0$ close to zero
This contradicts γ being the minimum, so this situation does not occur!

The minimizer E_1 has nonempty interior, $1 < p < n$

Here's $k(t) = \mathrm{Cap}_{\mathcal{A}}(E_1 + tE_2)^{-1/(n-p)} \sum_{i=1}^{m} c_i(\hat{q}_i + at)$
taking the derivative we get a term involving the derivative of the capacity which blows up approaching 0

$$\lim_{\tau \to 0} (p-1) \int_{\partial(E_1+\tau E_2)} h_2(g(x)) f(\nabla u(x)) dH^{n-1} = \infty$$

where g and u are Gauss and capacitary functions of $E_1 + \tau E_2$
This uses LN lower dimensional work

When $k = n-1$ we use the VV idea and get similarly that

$$\int_{\partial E} f(\nabla u_+) dH^{n-1} = \infty$$

u_+ means approaching from one side.

$E_1 + tE_2$, $k < n-1$

LN $(1-U) \geq ct^{\psi}$, $\psi = \frac{p-(n-k)}{p-1}$, at the $2t$ points, on a surface ball

$$\int_{\Delta_t} f(\nabla U) dH^{n-1} \geq c \left(\frac{1-U}{t} \right)^p t^{n-1} \geq c\, t^{p(\psi-1)+n-1}$$

There are about t^{-k} balls, summing over these

$$\sum_{\text{balls}} \int_{\Delta_t} f(\nabla U) dH^{n-1} \geq c\, t^{p(\psi-1)+n-1-k}$$

arithmetic

$$\sum_{\text{balls}} \int_{\Delta_t} f(\nabla U) dH^{n-1} \geq c\, t^{(k-(n-1))/(p-1)}$$

This is a negative exponent, let $t \to 0^+$.

$k = n - 1$, p-laplace argument

Krol', $(1 - U) \geq cv(x)$, where the "radial" part of v is $[(x_1^2 + x_n^2)^{1/2}]^{1-1/p}$, E into Whitney cubes Q, let $s = s(Q)$.

$$\int_Q |\nabla U|^p dH^{n-1} \geq c \left(\frac{s^{1-1/p}}{s} \right)^p s^{n-1} = cs^{n-2}$$

There are about $2^{l(n-2)}$ cubes of size about 2^{-l}, for l large, summing over these cubes gives a sum $\geq c$. Summing over all l gives infinity.

$k = n - 1$, VV

We have $B(0,1)$, E, E_t all in $B(0,1)$. then

$$t \int_E f(\nabla G_{E_t}) \geq c((F - G_E)(z_t) - (F - G_B)(z_t)) \geq c$$

divide by t let $t \to 0^+$.