Exponential decay estimates for fundamental solutions of Schrödinger-type operators

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The electric Schrödinger operator

By an electric Schrödinger operator we mean a second-order linear operator of the form

$$L := -\text{div} A \nabla + V,$$

where V is a scalar, real-valued, positive a.e., locally integrable function, and A is a matrix of bounded, measurable complex coefficients satisfying the uniform ellipticity condition

$$\lambda |\xi|^2 \leq \Re e \, \langle A(x)\xi, \xi \rangle \equiv \Re e \sum_{i,j=1}^{n} A_{ij}(x) \xi_j \bar{\xi_i} \text{ and } \quad \|A\|_{L^\infty(\mathbb{R}^n)} \leq \Lambda, \tag{1}$$

for some $\lambda > 0$, $\Lambda < \infty$ and for all $\xi \in \mathbb{C}^n$, $x \in \mathbb{R}^n$.

- The exponential decay of solutions to the Schrödinger operator in the presence of a positive potential is an important property underpinning foundation of quantum physics.
- However, establishing a precise rate of decay for complicated potentials is a challenging open problem to this date. (Landis conjecture)

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Exponential Decay of electric Schrödinger operators

• First results expressing upper estimates on the solutions in terms of a certain distance associated to V go back to Agmon [1], but not sharp.

• For eigenfunctions, the decay is governed by the uncertainty principle - see ADFJM '*Localization of eigenfunctions via an effective potential*'[2]

• In [9], Shen proved that if $V \in RH_{\frac{n}{2}}$, then the fundamental solution Γ to the classical Schrödinger operator $-\Delta + V$ satisfies the bounds

$$\frac{c_1 e^{-\varepsilon_1 d(x,y,V)}}{|x-y|^{n-2}} \leq \Gamma(x,y) \leq \frac{c_2 e^{-\varepsilon_2 d(x,y,V)}}{|x-y|^{n-2}}, \qquad (2)$$

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Exponential Decay of electric Schrödinger operators

• A natural question is whether the sharp exponential decay found by Shen for the fundamental solution to $-\Delta + V$ can be extended to the non self-adjoint setting $-\text{div } A\nabla + V$.

• Moreover, we also wondered whether we can obtain exponential decay results for the fundamental solution to the *magnetic Schrödinger operator*.

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• Moreover, we also wondered whether we can obtain exponential decay results for the fundamental solution to the *magnetic Schrödinger operator*.

The generalized magnetic Schrödinger operator

We consider the operator formally given by

$$L = -(\nabla - i\mathbf{a})^T A(\nabla - i\mathbf{a}) + V, \tag{3}$$

where $\mathbf{a} = (a_1, \ldots, a_n)$ is a vector of real-valued $L^2_{loc}(\mathbb{R}^n)$ functions, A and V as before. Denote

$$D_{\mathbf{a}} = \nabla - i\mathbf{a},$$

and the magnetic field by $\mathbf{B}$, so that

$$\mathbf{B} = \text{ curl } \mathbf{a}. \tag{4}$$

Properties of the magnetic Schrödinger operator

• The magnetic Schrödinger operator exhibits a property called *gauge invariance*: quantitative assumptions should be put on $\mathbf{B}$ rather than $\mathbf{a}$.

• The *diamagnetic inequality*

$$\Big|\nabla|u|(x)\Big| \leq \Big|D_{\mathbf{a}}u(x)\Big|. \tag{5}$$

• When $A \equiv I$ so that $L_M := L = (\nabla - i\mathbf{a})^2 + V$, the operator L_M is *dominated* by the Schrödinger operator $L_E := -\Delta + V$ in the following sense: for each $\varepsilon > 0$,

$$|(L_M + \varepsilon)^{-1} f| \leq (-\Delta + \varepsilon)^{-1}|f|, \qquad \text{for each } f \in H = L^2(\mathbb{R}^n). \tag{6}$$

The above is known as the *Kato-Simon inequality*.

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Existence of a fundamental solution to L_M?

• Some authors (Ben Ali [3], Kurata and Sugano [7]) showed properties of the fundamental solution to L_M under ad-hoc smoothness assumptions on the magnetic potential $\mathbf{a}$.

• On the other hand, the natural setting to make sense of L_M in the weak sense requires only that $\mathbf{a} \in L^2_{loc}(\mathbb{R}^n), V \in L^1_{loc}(\mathbb{R}^n)$.

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Some more history

• Before Shen obtained the sharp exponential decay result for the fundamental solution to $-\Delta + V$, he first obtained a polynomial decay result in [11].

• He later also obtained polynomial decay results for the magnetic Schrödinger operator L_M in [12].

• In [6], Kurata obtained exponential decay results for $-\text{div}\, A\nabla + V$ and L_M by integrating certain heat kernel estimates. He obtained the bound

$$|\Gamma(x,y)| \leq \frac{Ce^{-\varepsilon(1+m(x,V+|\mathbf{B}|)|x-y|)^{\frac{2}{2k_0+3}}}}{|x-y|^{n-2}} \quad \text{for a.e. } x,y \in \mathbb{R}^n,$$

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The Reverse Hölder class RH_p

To state our results, we need just a few more definitions.
We say that $w \in L^p_{loc}(\mathbb{R}^n)$, with $w > 0$ a.e., belongs to the Reverse Hölder class $RH_p = RH_p(\mathbb{R}^n)$ if there exists a constant C so that for any ball $B \subset \mathbb{R}^n$,

$$\left(\fint_B w^p \right)^{1/p} \leq C \fint_B w. \qquad (7)$$

The Fefferman-Phong-Shen maximal function $m(x, w)$

For a function $w \in RH_p, p \geq \frac{n}{2}$, define the maximal function $m(x, w)$ by

$$\frac{1}{m(x,w)} := \sup_{r>0}\Big\{r : \frac{1}{r^{n-2}} \int_{B(x,r)} w \leq 1\Big\}, \tag{8}$$

and the distance function

$$d(x, y, w) = \inf_{\gamma} \int_0^1 m(\gamma(t), w)|\gamma'(t)|\, dt, \tag{9}$$

where $\gamma : [0, 1] \to \mathbb{R}^n$ is absolutely continuous and $\gamma(0) = x, \gamma(1) = y$.

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$m(x, w)$ and the uncertainty principle

The function m measures the sum of the contributions of the kinetic energy $\Re e A D_{\mathbf{a}} f \overline{D_{\mathbf{a}} f}$ and potential energy $V|f|^2$, and is intimately related to the uncertainty principle through the following estimate which is often known as the *Fefferman-Phong* inequality:

Suppose that $\mathbf{a} \in L^2_{loc}(\mathbb{R}^n)^n$, and moreover assume (12) (next slide). Then, for all $u \in C^1_c(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} m^2(x, V + |\mathbf{B}|)|u|^2 \, dx \le C \int_{\mathbb{R}^n} (|D_{\mathbf{a}} u|^2 + V|u|^2) \, dx, \qquad (10)$$

where C depends on the constants c, c' from (12) and on $\|V + |\mathbf{B}|\|_{RH_{\frac{n}{2}}}$.

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L^2 exponential decay

Theorem 1 (Mayboroda, P. 2018)

For any operator L given by (3) and for any $f \in L^2(\mathbb{R}^n)$ with compact support, there exist constants $\tilde{d}, \varepsilon, C > 0$ such that

$$\int_{\left\{x\in\mathbb{R}^n \,|\, d(x,\operatorname{supp} f, V+|\mathbf{B}|)\geq \tilde{d}\right\}} m\left(\cdot, V+|\mathbf{B}|\right)^2 |u|^2 e^{2\varepsilon d(\cdot,\operatorname{supp} f, V+|\mathbf{B}|)} \leq C \int_{\mathbb{R}^n} |f|^2 \frac{1}{m(x,V+|\mathbf{B}|)^2}, \quad (11)$$

where $u := L^{-1}f$ (in a weak sense), provided that A is an elliptic matrix with complex bounded measurable coefficients, and

i) *either $\mathbf{a} = 0$ and $V \in RH_{n/2}$,*

ii) *or, more generally, $\mathbf{a} \in L^2_{loc}(\mathbb{R}^n)$, $V > 0$ a.e. on $\mathbb{R}^n$, and*

$$\begin{cases} V+|\mathbf{B}| \in RH_{n/2}, \\ 0 \leq V \leq c\, m(\cdot, V+|\mathbf{B}|)^2, \\ |\nabla \mathbf{B}| \leq c'\, m(\cdot, V+|\mathbf{B}|)^3. \end{cases} \quad (12)$$

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L^2 exponential decay for the resolvent

An analogous estimate holds for the resolvent operator $(I + t^2 L)^{-1}$, $t > 0$:

$$\int\limits_{\left\{x \in \mathbb{R}^n \,|\, d(x, \mathrm{supp}\, f, \mathscr{B}_t) \geq \tilde{d}\right\}} m\Big(\cdot, \mathscr{B}_t\Big)^2 \left|(I + t^2 L)^{-1} f\right|^2 e^{2\varepsilon d(\cdot, \mathrm{supp}\, f, \mathscr{B}_t)}$$

$$\leq C \int\limits_{\mathbb{R}^n} |f|^2 m\Big(\cdot, \mathscr{B}_t\Big)^2.$$

where $\mathscr{B} := V + |\mathbf{B}| + \frac{1}{t^2}$.

- In other words, $L^{-1} f$ decays as $e^{-\varepsilon d(\cdot, \mathrm{supp}\, f, V + |\mathbf{B}|)}$ away from the support of f and the resolvent decays as $e^{-\varepsilon d(\cdot, \mathrm{supp}\, f, V + |\mathbf{B}| + \frac{1}{t^2})}$.

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• L^2 exponential decay for the resolvents has appeared in the literature, see Germinet and Klein [5], but purely in terms of $\frac{1}{t^2}$. Also in many other sources.

• The estimate (11) (for the operator L^{-1}) is entirely new and is a consequence of the decay afforded by our assumptions on V and $\mathbf{B}$.

• Our results are in the nature of best possible under the very general assumptions.

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• Our results are in the nature of best possible under the very general assumptions.

A Moser estimate

Definition 2

We say that the operator L has a *Moser estimate* if for each ball $B \subset \mathbb{R}^n$ and each function u which solves $Lu = 0$ in the weak sense on $2B$, it follows that $u \in L^\infty(B)$ and

$$\|u\|_{L^\infty(cB)} \leq C\Big(\fint_{2B} |u|^2\Big)^{\frac{1}{2}}, \tag{13}$$

where c, C are independent of B and u.

- The electric Schrödinger operators with real matrix A, and the magnetic Schrödinger operator both have Moser estimates.

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Upper bound exponential decay

Theorem 3 (Mayboroda, P. 2018)

Suppose $\mathbf{a} \in L^2_{loc}(\mathbb{R}^n)$, A *is an elliptic matrix with complex, bounded coefficients,* $V \in L^1_{loc}(\mathbb{R}^n)$, *and that* L *is an operator for which there exists a fundamental solution bounded above by a multiple of* $|x-y|^{2-n}$. *Moreover, if* $\mathbf{a} \equiv 0$, *assume* $V \in RH_{\frac{n}{2}}$; *otherwise assume (12). Then there exists* $\varepsilon > 0$ *and a constant* $C > 0$, *depending on* L, *such that*

$$\left(\fint_{B(x, \frac{1}{m(x, V+|\mathbf{B}|)})} |\Gamma(z,y)|^2 \, dz \right)^{\frac{1}{2}} \leq \frac{C e^{-\varepsilon d(x,y,V+|\mathbf{B}|)}}{|x-y|^{n-2}} \text{ for all } x, y \in \mathbb{R}^n. \tag{14}$$

If L *has a Moser estimate, then*

$$|\Gamma(x,y)| \leq \frac{C e^{-\varepsilon d(x,y,V+|\mathbf{B}|)}}{|x-y|^{n-2}} \text{ for all } x, y \in \mathbb{R}^n. \tag{15}$$

Upper bound exponential decay

Theorem 3 establishes in particular the upper bound exponential decay for both electric Schrödinger operators $-\text{div } A\nabla + V$, and the magnetic Schrödinger operator $(\nabla - i\mathbf{a})^2 + V$.

A scale-invariant Harnack inequality

Definition 4

We say that the operator L satisfying assumptions (12) has the m*-scale invariant Harnack Inequality* if whenever $B = B(x_0, r)$, $r < \frac{c}{m(x_0, V+|\mathbf{B}|)}$, $x_0 \in \mathbb{R}^n$, the following property holds. For any u which solves $Lu = 0$ in the weak sense on $2B$,

$$\sup_{x\in B} |u(x)| \leq C \inf_{x\in B} |u(x)|, \tag{16}$$

with the constant $C > 0$ independent of B.

- The electric Schrödinger operators with real matrix A have the m-scale invariant Harnack inequality.

A scale-invariant Harnack inequality

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We say that the operator L satisfying assumptions (12) has the *m-scale invariant Harnack Inequality* if whenever $B = B(x_0, r)$, $r < \frac{c}{m(x_0, V+|\mathbf{B}|)}$, $x_0 \in \mathbb{R}^n$, the following property holds. For any u which solves $Lu = 0$ in the weak sense on $2B$,

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• The electric Schrödinger operators with real matrix A have the m-scale invariant Harnack inequality.

Lower bound exponential decay

Theorem 5 (Mayboroda, P. 2018)

Suppose that $\mathbf{a} \in L^2_{loc}(\mathbb{R}^n)$*,* A *is an elliptic matrix with complex, bounded coefficients,* $V \in L^1_{loc}(\mathbb{R}^n)$*, and that* L*,* $L_0 := L - V$*,* L_0^* *are operators for which there exist fundamental solutions* $\Gamma \equiv \Gamma_V$*,* Γ_0 Γ_0^**. Assume that* $\Gamma_V, \Gamma_0, \Gamma_0^*$ *are bounded above by a multiple of* $|x-y|^{2-n}$*, and that* Γ_0 *is bounded below by a multiple of* $|x-y|^{2-n}$*. Suppose that* L *has a Moser estimate, and that* L *satisfies the* m*-scale invariant Harnack Inequality. Moreover, if* $\mathbf{a} \equiv 0$*, assume that* $V \in RH_{\frac{n}{2}}$*; otherwise assume (12). Then there exist constants* c *and* ε_2 *depending on* $\lambda, \Lambda, \|V + |\mathbf{B}|\|_{RH_{\frac{n}{2}}}, n$ *and the constants from (12) such that*

$$|\Gamma(x,y)| \geq \frac{c e^{-\varepsilon_2 d(x,y,V+|\mathbf{B}|)}}{|x-y|^{n-2}}. \tag{17}$$

Exponential decay of the real electric Schrödinger operators

If Γ_E is the fundamental solution to an electric Schrödinger operator with real matrix A and $V \in RH_{\frac{n}{2}}$, then the last few theorems imply that

$$\frac{c_1 e^{-\varepsilon_1 d(x,y,V)}}{|x-y|^{n-2}} \leq \Gamma(x,y) \leq \frac{c_2 e^{-\varepsilon_2 d(x,y,V)}}{|x-y|^{n-2}}. \tag{18}$$

Thanks for listening!

Thank you. :)

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