Regularity results for a penalized boundary obstacle problem

Donatella Danielli

Purdue University

AMS Sectional Meeting
Northeastern University
April 22, 2018

Thank you for the invitation!

In this talk we will discuss a two-penalty boundary obstacle problem of interest in thermics and fluid dynamics.

Our goal is to establish existence, uniqueness and optimal regularity of the solutions, as well as structural properties of the free boundary. The study hinges on the monotone character of a perturbed frequency function of Almgren's type, and the analysis of the associated blow-ups.

This is joint work with Thomas Backing and Rohit Jain.

Outline

- Motivation
- Statement of the problem and regularity results
- Monotonicity formulas and the study of the free boundary
- Future directions

The Signorini Problem

A problem in linear elasticity, first proposed by Signorini in 1959, was one of the driving forces in the study of Variational Inequalities. In its original formulation, it consists of finding the elastic equilibrium configuration of an anisotropic non-homogeneous elastic body, resting on a rigid frictionless surface and subject only to its mass forces.

The existence and uniqueness of solutions was proved by Fichera in 1963.

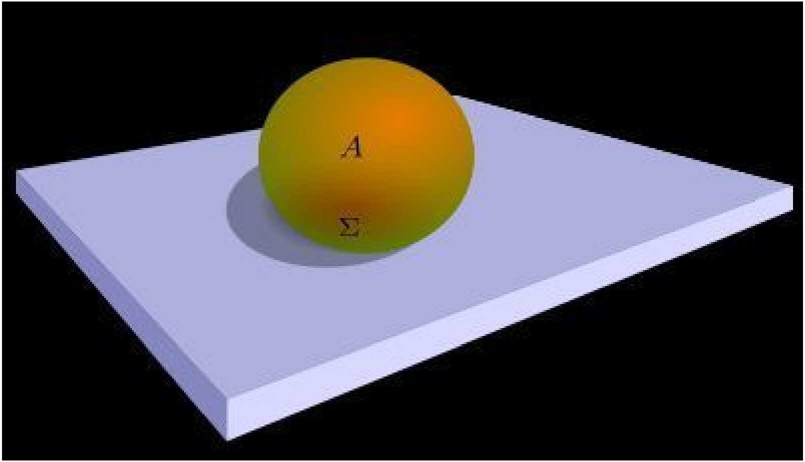
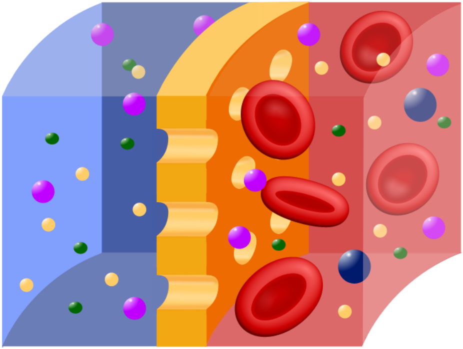


Figure: What will be the equilibrium configuration of an elastic body resting on a rigid frictionless plane?

Other applications include optimal control of temperature across a surface, in the modeling of semipermeable membranes where some saline concentration can flow through the membrane only in one direction, and financial math (when the random variation of underlying asset changes in a discontinuous fashion, as a Levi process).

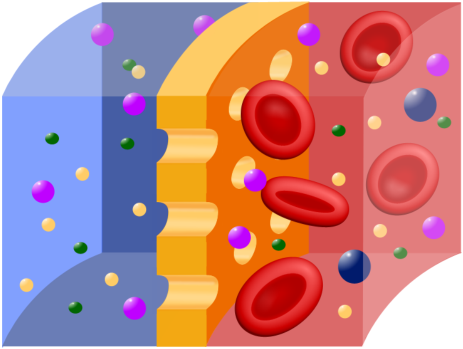
Semipermeable Membranes and Osmosis



Picture Source: Wikipedia

- **Semipermeable membrane** is a membrane that is permeable only for a certain type of molecules (*solvents*) and blocks other molecules (*solutes*).

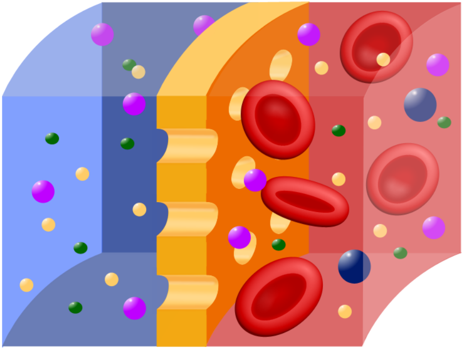
Semipermeable Membranes and Osmosis



Picture Source: Wikipedia

- **Semipermeable membrane** is a membrane that is permeable only for a certain type of molecules (*solvents*) and blocks other molecules (*solutes*).
- Because of the chemical imbalance, the solvent flows through the membrane from the region of smaller concentration of solute to the region of higher concentration (*osmotic pressure*).

Semipermeable Membranes and Osmosis

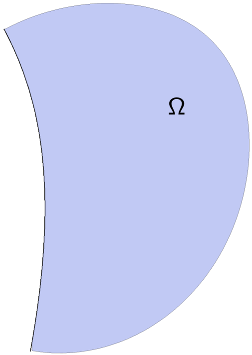


Picture Source: Wikipedia

- **Semipermeable membrane** is a membrane that is permeable only for a certain type of molecules (*solvents*) and blocks other molecules (*solutes*).
- Because of the chemical imbalance, the solvent flows through the membrane from the region of smaller concentration of solute to the region of higher concentration (*osmotic pressure*).
- The flow occurs in one direction. The flow continues until a sufficient pressure builds up on the other side of the membrane (to compensate for osmotic pressure), which then shuts the flow. This process is known as **osmosis**.

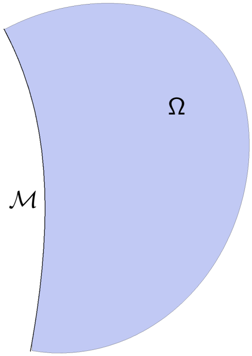
Mathematical Formulation

- Given open $\Omega \subset \mathbb{R}^n$



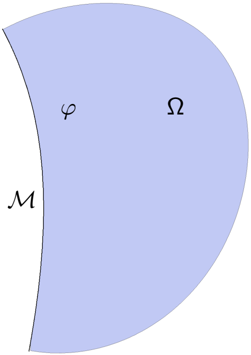
Mathematical Formulation

- Given open $\Omega \subset \mathbb{R}^n$
- $\mathcal{M} \subset \partial\Omega$ *semipermeable* part of the boundary



Mathematical Formulation

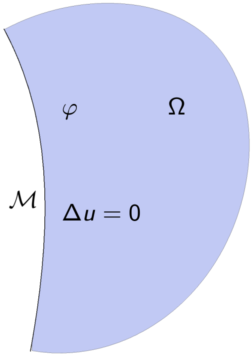
- Given open $\Omega \subset \mathbb{R}^n$
- $\mathcal{M} \subset \partial\Omega$ *semipermeable* part of the boundary
- $\varphi : \mathcal{M} \to \mathbb{R}$ *osmotic pressure*



Mathematical Formulation

- Given open $\Omega \subset \mathbb{R}^n$
- $\mathcal{M} \subset \partial\Omega$ *semipermeable* part of the boundary
- $\varphi : \mathcal{M} \to \mathbb{R}$ *osmotic pressure*
- $u : \Omega =\to \mathbb{R}$ *pressure* of the chemical solution, that satisfies the equation

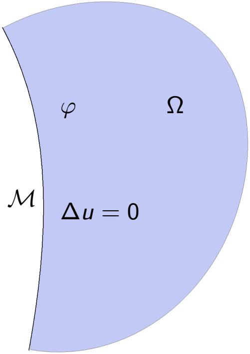
$$\Delta u = 0 \quad \text{in } \Omega$$



Mathematical Formulation

- Given open $\Omega \subset \mathbb{R}^n$
- $\mathcal{M} \subset \partial\Omega$ *semipermeable* part of the boundary
- $\varphi : \mathcal{M} \to \mathbb{R}$ *osmotic pressure*
- $u : \Omega =\!\!\to \mathbb{R}$ *pressure* of the chemical solution, that satisfies the equation

$$\Delta u = 0 \quad \text{in } \Omega$$



- We distinguish two cases.

Wall of Negligible Thickness

The boundary $\mathcal{M}$ consists of a semi-permeable membrane of negligible thickness. It allows the fluid which enters Ω to pass freely but prevents all outflow of fluid.

Wall of Negligible Thickness

The boundary $\mathcal{M}$ consists of a semi-permeable membrane of negligible thickness. It allows the fluid which enters Ω to pass freely but prevents all outflow of fluid.

Two situations are possible for points $x \in \Omega$:

Wall of Negligible Thickness

The boundary $\mathcal{M}$ consists of a semi-permeable membrane of negligible thickness. It allows the fluid which enters Ω to pass freely but prevents all outflow of fluid.

Two situations are possible for points $x \in \Omega$:

- $\varphi(x) < u(x)$
  When the outside pressure $\varphi(x)$ is smaller than the inside pressure $u(x)$, the fluid tries to leave Ω, but the wall prevents it. Thus,

$$\frac{\partial u}{\partial \nu} = 0.$$

Wall of Negligible Thickness

The boundary $\mathcal{M}$ consists of a semi-permeable membrane of negligible thickness. It allows the fluid which enters Ω to pass freely but prevents all outflow of fluid.

Two situations are possible for points $x \in \Omega$:

- $\varphi(x) < u(x)$
  When the outside pressure $\varphi(x)$ is smaller than the inside pressure $u(x)$, the fluid tries to leave Ω, but the wall prevents it. Thus,

$$\frac{\partial u}{\partial \nu} = 0.$$

Wall of Negligible Thickness

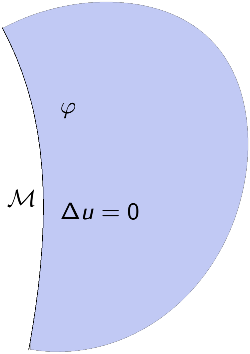
The boundary $\mathcal{M}$ consists of a semi-permeable membrane of negligible thickness. It allows the fluid which enters Ω to pass freely but prevents all outflow of fluid.

Two situations are possible for points $x \in \Omega$:

- $\varphi(x) < u(x)$
  When the outside pressure $\varphi(x)$ is smaller than the inside pressure $u(x)$, the fluid tries to leave Ω, but the wall prevents it. Thus,
  $$\frac{\partial u}{\partial \nu} = 0.$$
- $\varphi(x) \geq u(x)$
  In this case the wall allows the fluid to enter into Ω, so that $v \cdot \nu \leq 0$ (v denoting the velocity field).
  By Darcy's law $v = -K \nabla u$ ($K > 0$) and therefore
  $$\frac{\partial u}{\partial \nu} \geq 0.$$

- Since the flux must be finite, continuity considerations coupled with the negligible thickness of the wall imply $u = \varphi$ on $\mathcal{M}$. In conclusion, we

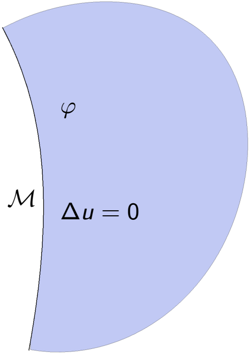
$$
\begin{aligned}
u &\geq \varphi \\
\partial_\nu u &\geq 0 \\
(u - \varphi)\partial_\nu u &= 0
\end{aligned}
$$



- Since the flux must be finite, continuity considerations coupled with the negligible thickness of the wall imply $u = \varphi$ on $\mathcal{M}$. In conclusion, we

$$\begin{aligned} u &\geq \varphi \\ \partial_\nu u &\geq 0 \\ (u-\varphi)\partial_\nu u &= 0 \end{aligned}$$

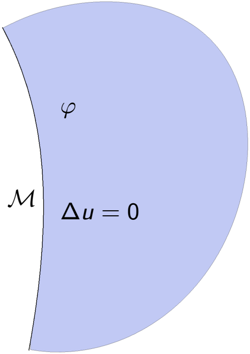
- These are known as the Signorini boundary conditions



- Since the flux must be finite, continuity considerations coupled with the negligible thickness of the wall imply $u = \varphi$ on $\mathcal{M}$. In conclusion, we

$$u \geq \varphi$$
$$\partial_\nu u \geq 0$$
$$(u - \varphi)\partial_\nu u = 0$$

- These are known as the Signorini boundary conditions
- Since u should stay above φ on $\mathcal{M}$, φ is known as the thin obstacle. The problem is also known as the Thin Obstacle Problem.



Although formulated in the 1960's, only in recent years there has been some significant progress on it.

Although formulated in the 1960's, only in recent years there has been some significant progress on it.

One of the main goals is to understand the properties of the coincidence set $\Lambda(u) := \{x \in \mathcal{M} : u = \varphi\}$ and its boundary (in the relative topology of $\mathcal{M}$) $\Gamma(u) := \partial_{\mathcal{M}}\Lambda(u)$, i.e., the free boundary. In order to do so, one needs to establish the optimal regularity of the solution across the free boundary.

Although formulated in the 1960's, only in recent years there has been some significant progress on it.

One of the main goals is to understand the properties of the coincidence set $\Lambda(u) := \{x \in \mathcal{M} : u = \varphi\}$ and its boundary (in the relative topology of $\mathcal{M}$) $\Gamma(u) := \partial_{\mathcal{M}}\Lambda(u)$, i.e., the free boundary. In order to do so, one needs to establish the optimal regularity of the solution across the free boundary.

When $\mathcal{M}$ and φ are smooth, Caffarelli proved in 1979 that the minimizer u in the thin obstacle problem is of class $C^{1,\alpha}_{\mathrm{loc}}(\Omega_\pm \cup \mathcal{M})$.

Normalization

Simplifying assumptions:

1. Vanishing thin obstacle φ.

2. The manifold $\mathcal{M}$ is a flat portion of the boundary of the relevant domain: $\mathcal{M}=\mathbb{R}^{n-1} \times \{0\}$.

Normalization

Simplifying assumptions:
1. Vanishing thin obstacle φ.
2. The manifold $\mathcal{M}$ is a flat portion of the boundary of the relevant domain: $\mathcal{M}=\mathbb{R}^{n-1} \times \{0\}$.

Since we are interested in properties of minimizers near free boundary points, after translation, rotation and scaling arguments we may consider a function u defined in the upper half-ball $B_1^+ := B_1 \cap \mathbb{R}^n_+$ satisfying

$$\Delta u = 0 \quad \text{in } B_1^+ \tag{0.1}$$

$$u \geq 0, \quad -\partial_{x_n} u \geq 0, \quad u\, \partial_{x_n} u = 0 \quad \text{on } B_1' \tag{0.2}$$

$$0 \in \Gamma(u) = \partial\Lambda(u) := \partial\{(x', 0) \in B_1' \mid u(x', 0) = 0\}, \tag{0.3}$$

where $\Lambda(u)$ is the coincidence set and the boundary is in the relative topology of B_1'. Here $B_1' := B_1 \cap (\mathbb{R}^{n-1} \times \{0\})$.

Recent Developments

- Athanasopoulos-Caffarelli (2006): Optimal $C^{1,1/2}$ interior regularity.

Recent Developments

- Athanasopoulos-Caffarelli (2006): Optimal $C^{1,1/2}$ interior regularity.
- Athanasopoulos-Caffarelli-Salsa (2008): Fine regularity properties of the free boundary. Namely, the set of regular free boundary points is locally a $C^{1,\alpha}$-manifold of dimension $n-2$.

Recent Developments

- Athanasopoulos-Caffarelli (2006): Optimal $C^{1,1/2}$ interior regularity.
- Athanasopoulos-Caffarelli-Salsa (2008): Fine regularity properties of the free boundary. Namely, the set of regular free boundary points is locally a $C^{1,\alpha}$-manifold of dimension $n-2$.
- In the particular case $\Omega = \mathbb{R}^{n-1} \times (0,\infty)$ and $\mathcal{M} = \mathbb{R}^{n-1} \times \{0\}$, the Signorini problem can be interpreted as an obstacle problem for the fractional Laplacian on $\mathbb{R}^{n-1}$:

$$u - \varphi \geq 0, \quad (-\Delta_{x'})^s u \geq 0, \quad (u-\varphi)(-\Delta_{x'})^s u = 0,$$

with $s = 1/2$.

Recent Developments

- Athanasopoulos-Caffarelli (2006): Optimal $C^{1,1/2}$ interior regularity.
- Athanasopoulos-Caffarelli-Salsa (2008): Fine regularity properties of the free boundary. Namely, the set of regular free boundary points is locally a $C^{1,\alpha}$-manifold of dimension $n-2$.
- In the particular case $\Omega = \mathbb{R}^{n-1} \times (0,\infty)$ and $\mathcal{M} = \mathbb{R}^{n-1} \times \{0\}$, the Signorini problem can be interpreted as an obstacle problem for the fractional Laplacian on $\mathbb{R}^{n-1}$:

$$u - \varphi \geq 0, \quad (-\Delta_{x'})^s u \geq 0, \quad (u-\varphi)(-\Delta_{x'})^s u = 0,$$

  with $s = 1/2$.
  - Silvestre (2007): Almost optimal regularity of solutions, namely $u \in C^{1,\alpha}(\mathbb{R}^{n-1})$ for any $\alpha < s$, $0 < s < 1$.
  - Caffarelli-Salsa-Silvestre (2008): Optimal regularity $C^{1,s}(\mathbb{R}^{n-1})$, free boundary regularity.

- Garofalo-Petrosyan (2009): Structure of the singular set of solutions to the thin obstacle problem by construction of two one-parameter families of monotonicity formulas (of Weiss and Monneau type).

- Garofalo-Petrosyan (2009): Structure of the singular set of solutions to the thin obstacle problem by construction of two one-parameter families of monotonicity formulas (of Weiss and Monneau type).
- Higher regularity of the free boundary around regular points:
  - De Silva-Savin (2014)
    C^∞ regularity (based on boundary Harnack estimates in slit domains)
  - Koch-Petrosyan-Shi (2014)
    Analiticity (based on a partial hodograph-Legendre transformation)

Wall of Finite Thickness

Again, two situations are possible for points $x \in \Omega$:

Wall of Finite Thickness

Again, two situations are possible for points $x \in \Omega$:

- $\varphi(x) < u(x)$
  When the outside pressure $\varphi(x)$ is smaller than the inside pressure $u(x)$, the fluid tries to leave Ω, but the wall prevents it. Thus,

$$\frac{\partial u}{\partial \nu} = 0.$$

Wall of Finite Thickness

Again, two situations are possible for points $x \in \Omega$:

- $\varphi(x) < u(x)$
  When the outside pressure $\varphi(x)$ is smaller than the inside pressure $u(x)$, the fluid tries to leave Ω, but the wall prevents it. Thus,

$$\frac{\partial u}{\partial \nu} = 0.$$

Wall of Finite Thickness

Again, two situations are possible for points $x \in \Omega$:

- $\varphi(x) < u(x)$
  When the outside pressure $\varphi(x)$ is smaller than the inside pressure $u(x)$, the fluid tries to leave Ω, but the wall prevents it. Thus,

$$\frac{\partial u}{\partial \nu} = 0.$$

- $\varphi(x) \geq u(x)$
  It is reasonable to assume that the outflow through the wall is proportional to the difference in pressure:

$$-\frac{\partial u}{\partial \nu} = k(u - \varphi),$$

where $k > 0$ measures the conductivity of the wall.

Remarks

- If the conductivity $k = 0$, no fluid enter or leaves the chamber and the pressure u is a solution to the Neumann problem.

Remarks

- If the conductivity $k = 0$, no fluid enter or leaves the chamber and the pressure u is a solution to the Neumann problem.

Remarks

- If the conductivity $k = 0$, no fluid enter or leaves the chamber and the pressure u is a solution to the Neumann problem.
- If the conductivity $k \to \infty$, in the limit one recovers the Signorini boundary conditions. Duvaut and Lions showed that if u_k is the solution corresponding to the conductivity k, then u_k converges weakly in L^2 to the solution to the thin obstacle problem.

Here we consider a local version of the problem, posed in the upper half ball

$$B_1^+ = \{x \in B_1 \mid x_n > 0\}.$$

Here we consider a local version of the problem, posed in the upper half ball

$$B_1^+ = \{x \in B_1 \mid x_n > 0\}.$$

Additionally, we will let $\varphi = 0$, but we will allow for fluid flow to occur both *into* and *out* of Ω with different permeability constants, under the assumption that the *flux in each direction is proportional to a power of the pressure*.

Here we consider a local version of the problem, posed in the upper half ball

$$B_1^+ = \{x \in B_1 \mid x_n > 0\}.$$

Additionally, we will let $\varphi = 0$, but we will allow for fluid flow to occur both *into* and *out* of Ω with different permeability constants, under the assumption that the *flux in each direction is proportional to a power of the pressure*.

Here we consider a local version of the problem, posed in the upper half ball

$$B_1^+ = \{x \in B_1 \mid x_n > 0\}.$$

Additionally, we will let $\varphi = 0$, but we will allow for fluid flow to occur both *into* and *out* of Ω with different permeability constants, under the assumption that the *flux in each direction is proportional to a power of the pressure*.

This allows an alternate interpretation of the problem as a boundary temperature control problem, as derived by Duvaut and Lions. The same model also describes the flux of electricity through semi-conducting walls.

Boundary temperature control

Assume that a continuous medium occupies a region Ω in $\mathbb{R}^n$, with boundary Γ and outer unit normal ν.

Given a reference temperature $h(x)$, for $x \in \Gamma$, it is required that the temperature at the boundary $u(x,t)$ deviates as little as possible from $h(x)$.

Thermostatic controls are placed on the boundary to inject an appropriate heat flux when necessary. The controls are regulated as follows:

Boundary temperature control

Assume that a continuous medium occupies a region Ω in $\mathbb{R}^n$, with boundary Γ and outer unit normal ν.

Given a reference temperature $h(x)$, for $x \in \Gamma$, it is required that the temperature at the boundary $u(x,t)$ deviates as little as possible from $h(x)$.

Thermostatic controls are placed on the boundary to inject an appropriate heat flux when necessary. The controls are regulated as follows:

- If $u(x,t) = h(x)$, no correction is needed and therefore the heat flux is null.
- If $u(x,t) \neq h(x)$, a quantity of heat proportional to the difference between $u(x,t)$ and $h(x)$ is injected.

The boundary condition can be written as

$$-\frac{\partial u}{\partial \nu} = \Phi(u),$$

where

$$\Phi(u) = \begin{cases} k_-(u-h) & \text{if } u < h \\ 0 & \text{if } u = h \\ k_+(u-h) & \text{if } u > h \end{cases}$$

Statement of the problem

In this setting the problem becomes

$$\begin{cases} \Delta u = 0 \text{ in } B_1^+ \\ u = g \text{ on } (\partial B_1)^+ \\ u_{x_n} = k_+(u^+)^{p-1} - k_-(u^-)^{p-1} \text{ on } \Gamma \end{cases}$$

where $g \in C^{2,\alpha}\left(\overline{B_1}\right)$ is the given boundary datum, $p > 1$, and

$$\begin{aligned} (\partial B_1)^+ &= \{x \in \partial B_1 \mid x_n > 0\} \\ \Gamma &= \{x \in B_1 \mid x_n = 0\} \\ u^+ &= \max\{u, 0\}, \ u^- = -\min\{u, 0\} \geq 0. \end{aligned}$$

Variational Formulation

We seek to minimize

$$J(v) = \frac{1}{2}\left(\int_{B_1} |\nabla v|^2 + \int_\Gamma \tilde{k}_-(v^-)^p + \int_\Gamma \tilde{k}_+(v^+)^p\right)$$

over all $v \in W^{1,2}(B_1)$, $v - g \in W_0^{1,2}(B_1)$ for a given boundary datum $g \in C^{2,\alpha}\left(\overline{B_1}\right)$.

Variational Formulation

We seek to minimize

$$J(v) = \frac{1}{2}\left(\int_{B_1} |\nabla v|^2 + \int_\Gamma \tilde{k}_-(v^-)^p + \int_\Gamma \tilde{k}_+(v^+)^p\right)$$

over all $v \in W^{1,2}(B_1),\ v - g \in W_0^{1,2}(B_1)$ for a given boundary datum $g \in C^{2,\alpha}\left(\overline{B_1}\right)$.

In this context we think of the data g as extended to all of B_1 by even reflection. A minimizer will be symmetric about Γ and u will correspond to its restriction to B_1^+.

Variational Formulation

We seek to minimize

$$J(v) = \frac{1}{2}\left(\int_{B_1} |\nabla v|^2 + \int_\Gamma \tilde{k}_-(v^-)^p + \int_\Gamma \tilde{k}_+(v^+)^p\right)$$

over all $v \in W^{1,2}(B_1),\ v - g \in W_0^{1,2}(B_1)$ for a given boundary datum $g \in C^{2,\alpha}\left(\overline{B_1}\right)$.

In this context we think of the data g as extended to all of B_1 by even reflection. A minimizer will be symmetric about Γ and u will correspond to its restriction to B_1^+.

Note: $\tilde{k}_\pm = 2k_\pm/p$.

- Allen-Lindgren-Petrosyan (2015)
  Studied minimizers of

$$J_a(v) = \int_{B_1^+} |\nabla v|^2 x_n^a + 2\int_\Gamma \left(k_-(v^-)^1 + k_+(v^+)^1\right)$$

with $a \in (-1, 1)$.
Proved optimal regularity of the minimizer u: For $K \Subset B^+ \cup \Gamma$

$$u \in C^{0,1-a}(K) \qquad \text{if } a \geq 0,$$

$$u \in C^{1,-a}(K) \qquad \text{if } a < 0,$$

as well as separation of the two free boundaries $\partial\{u > 0\} \cap \Gamma$ and $\partial\{u < 0\} \cap \Gamma$ when $a \geq 0$.

- Allen (2016)
  Considered the problem

$$\operatorname{div}\left(x_n^a \nabla u(x', x_n)\right) = 0 \text{ in } B_1^+,$$
$$\lim_{x_n \to 0} x_n^a u_{x_n}(x', x_n) = -ku^+(x, 0) \text{ on } \Gamma,$$

  with $k > 0$.
  The main objective is the study of the singular points of the free boundary.

Comparison with Signorini problem

New difficulties:

1. Non-homogeneous boundary condition $\Rightarrow$ this problem does not admit global homogeneous solutions of any degree.
   Existence and classification of such solutions play a pivotal role in the Signorini problem.

Comparison with Signorini problem

New difficulties:

1. Non-homogeneous boundary condition $\Rightarrow$ this problem does not admit global homogeneous solutions of any degree.
   Existence and classification of such solutions play a pivotal role in the Signorini problem.
2. In the thin obstacle problem continuity arguments force $u \geq h$, but the case $h > u$ is no longer ruled out when considering walls of finite thickness.
   Allowing for both constants $k^+, \ k^-$ to be finite (even when one of the two vanishes) de facto destroys the one-phase character of the problem.

Comparison with Signorini problem

New difficulties:

1. Non-homogeneous boundary condition $\Rightarrow$ this problem does not admit global homogeneous solutions of any degree.
   Existence and classification of such solutions play a pivotal role in the Signorini problem.
2. In the thin obstacle problem continuity arguments force $u \geq h$, but the case $h > u$ is no longer ruled out when considering walls of finite thickness.
   Allowing for both constants $k^+,\ k^-$ to be finite (even when one of the two vanishes) de facto destroys the one-phase character of the problem.

Redeeming feature:

The non-homogeneous character of the boundary condition allows to employ bootstrap arguments to prove regularity.

Existence and Uniqueness

Theorem 1

There exists a unique minimizer $u \in \{v \in W^{1,2}(B_1) \mid v - g \in W_0^{1,2}(B_1)\}$ *for the energy* $J(v)$.

Existence and Uniqueness

Theorem 1

There exists a unique minimizer $u \in \{v \in W^{1,2}(B_1) \mid v - g \in W_0^{1,2}(B_1)\}$ *for the energy* $J(v)$.

Theorem 2

The minimizer u *is a weak solution, i.e.,*

$$\int_{B_1^+} \nabla u \nabla \xi = -\int_{\Gamma} (-k_-(u^-)^{p-1} + k_+(u^+)^{p-1})\xi \tag{0.4}$$

for all $\xi \in C^\infty(B_1^+)$ *vanishing on* $(\partial B_1)^+$.

Existence and Uniqueness

Theorem 1

There exists a unique minimizer $u \in \{v \in W^{1,2}(B_1) \mid v - g \in W_0^{1,2}(B_1)\}$ *for the energy* $J(v)$.

Theorem 2

The minimizer u *is a weak solution, i.e.,*

$$\int_{B_1^+} \nabla u \nabla \xi = -\int_{\Gamma} (-k_-(u^-)^{p-1} + k_+(u^+)^{p-1})\xi \qquad (0.4)$$

for all $\xi \in C^\infty(B_1^+)$ *vanishing on* $(\partial B_1)^+$.

Proofs: Standard variational arguments.

Regularity of solutions

Theorem 3

Let $g \in C^{2,\alpha}(\overline{B_1})$ and let $k_\pm$ be non-negative, finite and non-equal constants. Let u be the unique minimizer of the energy $J(v)$. Then

- $u \in C^{\lfloor p-1 \rfloor,\alpha}(\overline{B^+_{1/2}})$ *for every* $\alpha < p - 1 - \lfloor p-1 \rfloor$, *if* p *is not an integer.*
- $u \in C^{p-1,\alpha}(\overline{B^+_{1/2}})$ *for every* $\alpha < p - 1$, *if* p *is an integer.*

Additionally, if $k_- = k_+$ *or if* g *does not change sign, then* $u \in C^\infty(\overline{B^+_{1/2}})$.

Sketch of proof

Starting point: Energy estimate

Lemma 4

Let u be the minimizer to $J(v)$, $r > 0$. Then, for any $B_{2r} \subset B_1$

$$\int_{B_r} |\nabla u|^2 \,\mathrm{d}x \leq \frac{c}{r^2} \int_{B_{2r}} u^2 \,\mathrm{d}x.$$

Sketch of proof

Starting point: Energy estimate

Lemma 4

Let u be the minimizer to $J(v)$, $r > 0$. Then, for any $B_{2r} \subset B_1$

$$\int_{B_r} |\nabla u|^2 \, \mathrm{d}x \le \frac{c}{r^2} \int_{B_{2r}} u^2 \, \mathrm{d}x.$$

Intermediate regularity: Hölder modulus of continuity

Lemma 5

Let u be as in Lemma 4. Then

$$u \in C^{0,1/2}(\overline{B_{1/2}}).$$

Conclusion:

$$u \in C^{0,1/2}(\overline{B_{1/2}}) \Rightarrow u \in C^{0,1/2}(\Gamma)$$
$$\Rightarrow u_{x_n} = -k_-(u^-)^{p-1} + k_+(u^+)^{p-1} \in C^{0,\alpha}(\Gamma)$$

for $\alpha > 0$.

Conclusion:

$$u \in C^{0,1/2}(\overline{B_{1/2}}) \Rightarrow u \in C^{0,1/2}(\Gamma)$$
$$\Rightarrow u_{x_n} = -k_-(u^-)^{p-1} + k_+(u^+)^{p-1} \in C^{0,\alpha}(\Gamma)$$

for $\alpha > 0$.

Hence, u is the solution to an oblique derivative problem, with Hölder continuous boundary datum.

Conclusion:

$$u \in C^{0,1/2}(\overline{B_{1/2}}) \Rightarrow u \in C^{0,1/2}(\Gamma)$$
$$\Rightarrow u_{x_n} = -k_-(u^-)^{p-1} + k_+(u^+)^{p-1} \in C^{0,\alpha}(\Gamma)$$

for $\alpha > 0$.

Hence, u is the solution to an oblique derivative problem, with Hölder continuous boundary datum. Regularity theory implies

$$u \in C^{1,\alpha}(\Gamma) \Rightarrow u_{x_n} = -k_-(u^-)^{p-1} + k_+(u^+)^{p-1} \in C^{0,p-1}(\Gamma) \quad \text{if } p \leq 2,$$

(or $u_{x_n} = -k_-(u^-)^{p-1} + k_+(u^+)^{p-1}$ is differentiable with a Hölder modulus of continuity if $p > 2$.)
Repeated application of the regularity theory and iteration give the desired result.

Conclusion:

$$u \in C^{0,1/2}(\overline{B_{1/2}}) \Rightarrow u \in C^{0,1/2}(\Gamma)$$
$$\Rightarrow u_{x_n} = -k_-(u^-)^{p-1} + k_+(u^+)^{p-1} \in C^{0,\alpha}(\Gamma)$$

for $\alpha > 0$.

Hence, u is the solution to an oblique derivative problem, with Hölder continuous boundary datum. Regularity theory implies

$$u \in C^{1,\alpha}(\Gamma) \Rightarrow u_{x_n} = -k_-(u^-)^{p-1} + k_+(u^+)^{p-1} \in C^{0,p-1}(\Gamma) \quad \text{if } p \leq 2,$$

(or $u_{x_n} = -k_-(u^-)^{p-1} + k_+(u^+)^{p-1}$ is differentiable with a Hölder modulus of continuity if $p > 2$.)
Repeated application of the regularity theory and iteration give the desired result.

Finally, if g does not change sign, then u does not change sign either $\Rightarrow u^{\pm} = u$. Thus $u^{\pm}$ is as smooth as u, and the regularity result can be bootstrapped to show smoothness.

Consider the case $p = 2$. Then, our regularity result ensures $u \in C^{1,\alpha}$ for all $\alpha < 1$. Is this optimal?

Optimal regularity

Consider the case $p = 2$. Then, our regularity result ensures $u \in C^{1,\alpha}$ for all $\alpha < 1$. Is this optimal?

Theorem 6

If $\nabla u(0) \neq 0$, then $u \notin C^{1,1}(0)$.

Optimal regularity

Consider the case $p = 2$. Then, our regularity result ensures $u \in C^{1,\alpha}$ for all $\alpha < 1$. Is this optimal?

Theorem 6

If $\nabla u(0) \neq 0$, *then* $u \notin C^{1,1}(0)$.

Proof. Argument by contradiction, based on comparison principle and construction of suitable barrier.

Optimal regularity

Consider the case $p = 2$. Then, our regularity result ensures $u \in C^{1,\alpha}$ for all $\alpha < 1$. Is this optimal?

Theorem 6

If $\nabla u(0) \neq 0$, then $u \notin C^{1,1}(0)$.

Proof. Argument by contradiction, based on comparison principle and construction of suitable barrier.

Open question: What is the optimal regularity when $p > 2$?

Free boundary: Regular set

The regular set of the free boundary is defined as

$$\mathcal{R} = \{(x', 0) \in \Gamma \mid u(x', 0) = 0, \ \nabla u(x', 0) \neq 0\}$$

Free boundary: Regular set

The regular set of the free boundary is defined as

$$\mathcal{R} = \{(x',0) \in \Gamma \mid u(x',0) = 0,\ \nabla u(x',0) \neq 0\}$$

Theorem 7

If $x_0' \in \mathcal{R}$, then in a neighborhood of x_0', the free boundary $\{u(x',0) = 0\}$ is a $C^{1,\alpha}-$ graph for all $\alpha < 1$.

Free boundary: Regular set

The regular set of the free boundary is defined as

$$\mathcal{R} = \{(x',0) \in \Gamma \mid u(x',0) = 0,\ \nabla u(x',0) \neq 0\}$$

Theorem 7

If $x_0' \in \mathcal{R}$, then in a neighborhood of x_0', the free boundary $\{u(x',0) = 0\}$ is a $C^{1,\alpha}-$ graph for all $\alpha < 1$.

Proof. Consequence of regularity result, and implicit function theorem.

The regular set of the free boundary is defined as

$$\mathcal{R} = \{(x',0) \in \Gamma \mid u(x',0) = 0,\ \nabla u(x',0) \neq 0\}$$

Theorem 7

If $x_0' \in \mathcal{R}$, then in a neighborhood of x_0', the free boundary $\{u(x',0) = 0\}$ is a $C^{1,\alpha}-$ graph for all $\alpha < 1$.

Proof. Consequence of regularity result, and implicit function theorem.

Open problem: Higher regularity of the free boundary.

A perturbed Almgren Frequency Functional

A crucial tool in the study of the Signorini problem is the Almgren's Frequency Functional

$$N(r,u) = \frac{r \int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2}$$

The name comes from fact that if u is a harmonic function in B_1, homogeneous of degree κ, then $N(r,u) = \kappa$.

Theorem 8 (Athanasopolous-Caffarelli-Salsa, 2007)

If u is a solution to the Signorini problem, then the function $N(r,u)$ is monotone increasing in r for $0 < r < 1$. Moreover, $N(r,u) \equiv \kappa$ for $0 < r < 1$ iff u is homogeneous of order κ in B_1, i.e.

$$x \cdot \nabla u - \kappa u = 0 \quad \text{in } B_1.$$

The Almgren's Frequency Functional is no longer monotone in our setting, but a suitable perturbation is.

The Almgren's Frequency Functional is no longer monotone in our setting, but a suitable perturbation is.

Theorem 9

Let $p \geq 2$, u be a solution, and let $F(u) = k_-(u^-)^p + k_+(u^+)^p$. Then the perturbed Almgren Frequency Functional

$$\tilde{N}(r,u) = r\frac{\int_{B_r^+} |\nabla u|^2 + \frac{2}{p}\int_\Gamma F(u)}{\int_{(\partial B_r)^+} u^2}$$

is monotone increasing in $r \in (0,1)$.

The Almgren's Frequency Functional is no longer monotone in our setting, but a suitable perturbation is.

Theorem 9

Let $p \geq 2$, u be a solution, and let $F(u) = k_-(u^-)^p + k_+(u^+)^p$. Then the perturbed Almgren Frequency Functional

$$\tilde{N}(r,u) = r\frac{\int_{B_r^+} |\nabla u|^2 + \frac{2}{p}\int_\Gamma F(u)}{\int_{(\partial B_r)^+} u^2}$$

is monotone increasing in $r \in (0,1)$.

Since $\tilde{N}(r,u) \geq 0$, we immediately have

Corollary 10

There exists $\lim_{r\to 0^+} \tilde{N}(r,u) = \mu \in [0,\infty)$.

Some consequences

To fix ideas, in the following we will always assume $p = 2$.
Recall

$$N(r,u) = \frac{r \int_{B_r^+} |\nabla u|^2}{\int_{\partial B_r^+} u^2}, \qquad F(u) = k_-(u^-)^2 + k_+(u^+)^2,$$

and

$$\tilde{N}(r,u) = r \frac{\int_{B_r^+} |\nabla u|^2 + \int_\Gamma F(u)}{\int_{(\partial B_r)^+} u^2} = N(r,u) + r \frac{\int_\Gamma F(u)}{\int_{(\partial B_r)^+} u^2}$$

Some consequences

To fix ideas, in the following we will always assume $p = 2$.
Recall

$$N(r,u) = \frac{r\int_{B_r^+} |\nabla u|^2}{\int_{\partial B_r^+} u^2}, \qquad F(u) = k_-(u^-)^2 + k_+(u^+)^2,$$

and

$$\tilde{N}(r,u) = r\frac{\int_{B_r^+} |\nabla u|^2 + \int_\Gamma F(u)}{\int_{(\partial B_r)^+} u^2} = N(r,u) + r\frac{\int_\Gamma F(u)}{\int_{(\partial B_r)^+} u^2}$$

Clearly $\tilde{N}(r,u) \geq N(r,u)$. Moreover, a Poincaré-type trace inequality implies

$$N(r,u) \geq \frac{\tilde{N}(r,u) - Cr}{1 + Cr}.$$

Some consequences

To fix ideas, in the following we will always assume $p = 2$.
Recall

$$N(r,u) = \frac{r\int_{B_r^+} |\nabla u|^2}{\int_{\partial B_r^+} u^2}, \qquad F(u) = k_-(u^-)^2 + k_+(u^+)^2,$$

and

$$\tilde{N}(r,u) = r\frac{\int_{B_r^+} |\nabla u|^2 + \int_\Gamma F(u)}{\int_{(\partial B_r)^+} u^2} = N(r,u) + r\frac{\int_\Gamma F(u)}{\int_{(\partial B_r)^+} u^2}$$

Clearly $\tilde{N}(r,u) \geq N(r,u)$. Moreover, a Poincaré-type trace inequality implies

$$N(r,u) \geq \frac{\tilde{N}(r,u) - Cr}{1 + Cr}.$$

Hence, there exists $\lim_{r\to 0^+} N(r,u) = \mu$.

From now assume $\nabla u(0) = 0$. We introduce

$$\varphi(r) = \varphi(r; u) = \fint_{(\partial B_r)^+} u^2.$$

Corollary 11

Let $0 \leq \lim_{r \to 0^+} \tilde{N}(r) = \mu < \infty$. *Then*
(a) The function $r \to r^{-2\mu}\varphi(r)$ *is nondecreasing for* $0 < r < 1$. *In particular,*

$$\varphi(r) \leq r^{2\mu}\varphi(1) \leq r^{2\mu} \sup_{B_1} |u|.$$

(b) Let $0 < r < 1$. $\forall \delta > 0$, $\exists r_0(\delta) > 0$ *such that* $\forall r, R \leq r_0(\delta)$,

$$\varphi(R) \leq \left(\frac{R}{r}\right)^{2(\mu+\delta)} \varphi(r).$$

Corollary 12

For all $x \in B_{r/2}$,

$$|u(x)| \leq r^{\mu} \sup_{B_1} |u|.$$

Corollary 12

For all $x \in B_{r/2}$,

$$|u(x)| \leq r^{\mu} \sup_{B_1} |u|.$$

The final step to obtain the optimal regularity estimate around free boundary points with vanishing gradient is to study blow-up sequences. Define

$$v_r(x) = \frac{u(rx)}{[\varphi(u,r)]^{1/2}}.$$

Note: $\|v_r\|_{L^2(\partial B_1)} = 1$, and as a consequence of the monotonicity of the perturbed Almgren Frequency Function and regularity estimates, $\{v_r\}$ are equibounded in H^1_{loc} and in $C^{1,\alpha}$.

Corollary 12

For all $x \in B_{r/2}$,

$$|u(x)| \leq r^{\mu} \sup_{B_1} |u|.$$

The final step to obtain the optimal regularity estimate around free boundary points with vanishing gradient is to study blow-up sequences. Define

$$v_r(x) = \frac{u(rx)}{[\varphi(u,r)]^{1/2}}.$$

Note: $\|v_r\|_{L^2(\partial B_1)} = 1$, and as a consequence of the monotonicity of the perturbed Almgren Frequency Function and regularity estimates, $\{v_r\}$ are equibounded in H^1_{loc} and in $C^{1,\alpha}$.

Thus, there exists a uniformly convergent subsequence on every compact subset of $\mathbb{R}^n$ such that $v_j \to v^*, \quad \nabla v_j \to \nabla v^*$.

Note: $\|v_r\|_{L^2(\partial B_1)} = 1 \Rightarrow$ the blow-up is nontrivial.
Moreover,

$$[u(rx)]_y = r u_{x_n}(rx) = r\left(k_+ u^+ - k_- u^-\right).$$

Note: $\|v_r\|_{L^2(\partial B_1)} = 1 \Rightarrow$ the blow-up is nontrivial.
Moreover,

$$[u(rx)]_y = ru_{x_n}(rx) = r\left(k_+u^+ - k_-u^-\right).$$

Letting $r \to 0$, we find that v^* satisfies

$$\begin{cases} \Delta v^* = 0 & \text{in } B_1^+. \\ v^*_{x_n} = 0 & \text{on } \Gamma. \end{cases} \tag{0.5}$$

As $r_j \to 0$,

$$\tilde{N}(r_j, u) = \tilde{N}(1, v_j) \to \tilde{N}(1, v^*) = \mu.$$

Hence v^* is homogenous of degree $\mu \geq 2$.

A monotonicity formula of Monneau type

We now introduce the Weiss functional

$$W_{\mu}(r, u) = \frac{H(r, u)}{r^{n-1+2\mu}}(N(r, u) - \mu),$$

where $H(r, u) = \int_{(\partial B_r)^+} u^2$.

A monotonicity formula of Monneau type

We now introduce the Weiss functional

$$W_\mu(r,u) = \frac{H(r,u)}{r^{n-1+2\mu}}(N(r,u)-\mu),$$

where $H(r,u) = \int_{(\partial B_r)^+} u^2$.

Let p_μ be a harmonic polynomial, homogeneous of degree μ and even in x_n. Using the estimates previously established, we can show

$$\frac{d}{dr}\left(\frac{1}{r^{n-1+2\mu}}\int_{(\partial B_r)^+}(u-p_\mu)^2\right) \geq \frac{2}{r}W(r,u) - C \geq -C'.$$

We have thus proved the quasi-monotonicity of the Monneau functional

$$\frac{1}{r^{n-1+2\mu}}\int_{(\partial B_r)^+}(u-p_\mu)^2$$

Consequences

- **Nondegeneracy**: There exists a constant $c > 0$ such that, for $r < 1$

$$\sup_{B_r^+} |u| \geq cr^2.$$

Consequences

- **Nondegeneracy**: There exists a constant $c > 0$ such that, for $r < 1$

$$\sup_{B_r^+} |u| \geq cr^2.$$

- **Uniqueness of the homogeneous blow-ups:** There exists a unique non-zero harmonic polynomial p_μ, homogeneous of degree $\mu \geq 2$ and even in x_n, such that

$$\tilde{v}_r(x) = \frac{u(rx)}{r^2} \to p_\mu(x).$$

Future developments

- Structure of the free boundary

Future developments

- Structure of the free boundary
- Separation of the two phases

Future developments

- Structure of the free boundary
- Separation of the two phases
- Parabolic case

Future developments

- Structure of the free boundary
- Separation of the two phases
- Parabolic case
- Refined analysis of the case $p > 2$

Future developments

- Structure of the free boundary
- Separation of the two phases
- Parabolic case
- Refined analysis of the case $p > 2$
- More general boundary condition: non-zero obstacle, gap in range for temperature controls...

Thank you for your attention!