The mixed boundary value problem in Lipschitz domains

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Classical boundary value problems

L^p **Dirichlet** and L^p **Regularity** problems:

$$(D)_p \quad \begin{cases} u \in \mathcal{C}^2(\Omega) \\ Lu = 0 \;\text{ in }\; \Omega \\ u|_{\partial\Omega} = f \in L^p(\partial\Omega) \\ u^* \in L^p(\partial\Omega) \end{cases} \qquad (R)_p \quad \begin{cases} u \in \mathcal{C}^2(\Omega) \\ Lu = 0 \;\text{ in }\; \Omega \\ u|_{\partial\Omega} = f \in L^p_1(\partial\Omega). \\ (\nabla u)^* \in L^p(\partial\Omega) \end{cases}$$

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L^p **Neumann** problem:

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- ν denotes the outward unit normal vector
- All boundary values in (D), (R), (N) are understood as non-tangential limits.

The mixed boundary value problem

Decompose $\partial\Omega = D \cup N$, $D \cap N = \emptyset$. Let $D \subset \partial\Omega$ be an open set (relative to $\partial\Omega$).

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L^p **mixed** problem:

$$(MP)_p \quad \begin{cases} u \in \mathcal{C}^2(\Omega) \\ Lu = 0 \quad \text{in} \quad \Omega \\ u\big|_D = f_D \in L^p_1(D) \\ \dfrac{\partial u}{\partial \nu}\big|_N = f_N \in L^p(N) \\ (\nabla u)^* \in L^p(\partial\Omega). \end{cases}$$

Boundary values in (MP) are understood as non-tangential limits.

The L^p mixed problem for the Laplacian

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain with boundary $\partial\Omega = D \cup N$, $D \cap N = \emptyset$.

$$(MP)_p \quad \begin{cases} \Delta u = 0 \quad \text{in} \quad \Omega \\ u\big|_D = f_D \in L^p_1(D) \\ \dfrac{\partial u}{\partial \nu}\big|_N = f_N \in L^p(N) \\ (\nabla u)^* \in L^p(\partial\Omega). \end{cases}$$

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- What conditions on Ω, N and D, and f_N and f_D ensure that a solution of $(MP)_p$ exists?
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- What conditions on Ω, N and D ensure that $(MP)_p$ has at most one solution?

Let $\Lambda = \partial D$ and let $\delta(x) = \text{dist}(x, \Lambda)$.

Corkscrew condition: There exists $M > 0$ such that for all $x \in \Lambda, 0 < r < r_0$, there exists $\tilde{x} \in D$ such that $|x - \tilde{x}| < r$ and $\delta(\tilde{x}) > M^{-1}r$.

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Theorem 1: L^1 result Brown, Ott & Taylor

Let D satisfy the corkscrew condition. If $f_N \in H^1(N)$ and $f_D \in H^{1,1}(D)$, the L^1-mixed problem has a solution that satisfies

$$\|(\nabla u)^*\|_{L^1(\partial\Omega)} \leq C \left(\|f_N\|_{H^1(N)} + \|f_D\|_{H^{1,1}(D)}\right).$$

Furthermore, the solution is unique in the class of functions with $(\nabla u)^* \in L^1(\partial\Omega)$.

The L^p mixed problem for the Laplacian

Theorem 2: L^p result Brown, Ott & Taylor

Let D satisfy the corkscrew condition. There exists an exponent $q_0 > 2$, depending on M and n, so that the following hold.

For p in the interval $(1, q_0/2)$, we have:

If $f_N \in L^p(N)$ and $f_D \in L^p_1(D)$ there exists a solution to the L^p-mixed problem. The solution u satisfies

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C \left(\|f_N\|_{L^p(N)} + \|f_D\|_{L^p_1(D)} \right).$$

Furthermore, the solution is unique in the class of functions satisfying $(\nabla u)^* \in L^p(\partial\Omega)$.

The L^p mixed problem for the Lamé system of elastostatics

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, $\partial\Omega = D \cup N$, $D \cap N = \emptyset$.

$$(MP)_p \quad \begin{cases} \mathcal{L}\vec{u} = \vec{0} \quad \text{in } \Omega \\ \vec{u}\big|_D = \vec{f}_D \in L^p_1(D) \\ \dfrac{\partial \vec{u}}{\partial \rho^s}\big|_N = \vec{f}_N \in L^p(N) \\ (\nabla u)^* \in L^p(\partial\Omega). \end{cases}$$

The action of $\mathcal{L}$ on $\vec{u}$ is given by

$$\mathcal{L}\vec{u} := \mu\Delta\vec{u} + (\lambda + \mu)\nabla \operatorname{div} \vec{u},$$

where $\lambda, \mu \in \mathbb{R}$ are called the Lamé moduli and satisfy

$$\mu > 0 \qquad \text{and} \qquad \lambda + \mu > 0.$$

The Lamé system of elastostatics

For each $s \in \mathbb{R}$, the tensor coefficient associated with $\mathcal{L}$ is understood to be the collection

$$A(s) := \left(a^{ij}_{\alpha\beta}(s)\right)_{i,j,\alpha,\beta \in \{1,2\}}$$

where

$$a^{ij}_{\alpha\beta}(s) := \mu\delta_{ij}\delta_{\alpha\beta} + (\lambda + \mu - s)\delta_{i\alpha}\delta_{j\beta} + s\delta_{i\ell}\delta_{jk},$$

$$\text{for all } i, j, \alpha, \beta \in \{1, 2\}.$$

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Neumann-type boundary conditions:

$$\left(\frac{\partial \vec{u}}{\partial \rho^s}\right)^\alpha = \nu_i a^{ij}_{\alpha\beta}(s) \frac{\partial u^\beta}{\partial x_j}.$$

Above, $\nu = (\nu_1, \nu_2)$ denotes the outward unit normal vector defined a.e. on $\partial\Omega$.

The L^p mixed problem for the Lamé system

Assume that $D \subset \partial\Omega$ satisfies the corkscrew condition.

Theorem 3: The L^p mixed problem for Lamé
Brown and Ott

1. There exists $p_0 > 1$ such that for $1 < p < p_0$, the L^p-mixed problem is well-posed in the following sense: if $f_N \in L^p(N)$, $f_D \in L^p_1(\partial\Omega)$, the L^p-mixed problem has a unique solution u satisfying

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C\big(\|f_N\|_{L^p(N)} + \|f_D\|_{L^p_1(\partial\Omega)}\big).$$

The boundary values of u exist as nontangential limits.

continued

The L^p mixed problem for the Lamé system

Theorem 3: The L^p mixed problem for Lamé

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2. There exists $p_1 < 1$ such that if $p_1 < p \leq 1$, the L^p-mixed problem is well-posed in the following sense: if $f_N \in H^p(N)$, $f_D \in H^{1,p}(\partial\Omega)$, the L^p-mixed problem has a unique solution u satisfying

$$\|u\|_{H^{1,p}(\partial\Omega)} + \|\frac{\partial u}{\partial \rho}\|_{H^p(\partial\Omega)} + \|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C\big(\|f_N\|_{H^p(N)} + \|f_D\|_{H^{1,p}(\partial\Omega)}\big).$$

Next steps

1. Well-posedness of mixed problem for the Stokes system of hydrostatics:

$$(MP) \quad \begin{cases} -\Delta u + \nabla p = f \quad \text{in} \quad \Omega \\ -\text{div} u = g \quad \text{in} \quad \Omega \\ u = f_D \quad \text{on} \quad D \\ 2\nu\epsilon(u) - p\nu = f_N \quad \text{on} \quad N. \end{cases}$$

   - $u : \Omega \to \mathbb{R}^2$, $p : \Omega \to \mathbb{R}$
   - $\epsilon(u)$ denotes the symmetric part of the gradient of u,

$$\epsilon_i^\alpha(u) = \frac{1}{2}\left(\frac{\partial u^\alpha}{x_i} + \frac{\partial u^i}{x_\alpha}\right).$$

Next steps

2. *Sharp* well-posedness of $(MP)_p$ for the Laplacian.

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A first step in this direction is to consider the L^p mixed problem for the Laplacian when the domain Ω is an infinite sector in $\mathbb{R}^2$, with a Dirichlet boundary condition imposed on one ray of the sector and a Neumann boundary condition is imposed on the other ray.

Well-posedness of $(MP)_p$ in a sector

- Seek a solution to $(MP)_p$, $p \in (1, \infty)$, expressed as a harmonic single layer potential operator with L^p density.

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- Seek a solution to $(MP)_p$, $p \in (1, \infty)$, expressed as a harmonic single layer potential operator with L^p density.

- This leads to the issue of inverting the operator

$$\begin{pmatrix} \partial_\tau S & \partial_\tau S \\ K^* & -\frac{1}{2}I + K^* \end{pmatrix} : L^p(D) \oplus L^p(N) \to L^p(D) \oplus L^p(N).$$

  - S is the boundary-to-boundary harmonic single layer potential operator,
  - ∂_τ denotes differentiation in the tangential direction,
  - K^* is the formal adjoint of the boundary-to-boundary harmonic double layer potential operator,
  - I is the identity operator.

Well-posedness of $(MP)_p$ in a sector

Theorem 4. Awala, I. Mitrea & Ott (2016)

Let $\Omega \subseteq \mathbb{R}^2$ be the interior of an infinite angle of aperture $\theta \in (0, 2\pi)$. Denote by $D := (\partial\Omega)_1$ and $N := (\partial\Omega)_2$ the left and right rays, respectively, of Ω. Then $(MP)_p$ for the Laplacian is well-posed whenever

$$p \neq \begin{cases} \frac{2\pi-\theta}{\pi-\theta} & \text{if } \theta \in (0, \pi/2], \\ \frac{2\pi-\theta}{\pi-\theta}, \frac{2\theta}{2\theta-\pi} & \text{if } \theta \in (\pi/2, \pi) \\ 2 & \text{if } \theta = \pi \\ \frac{2\theta}{2\theta-\pi}, \frac{\theta}{\theta-\pi} & \text{if } \theta \in (\pi, 3\pi/2] \\ \frac{2\theta}{2\theta-\pi}, \frac{2\theta}{2\theta-3\pi}, \frac{\theta}{\theta-\pi} & \text{if } \theta \in (3\pi/2, 2\pi). \end{cases}$$

The mixed problem in creased domains

- Brown and Sykes (1994, 2001) establish the well-posedness of $(MP)_p$ for the Laplacian for $p \in (1, 2]$ in the class of *creased* domains.
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- Via perturbation, the result is that for each bounded creased Lipschitz domain, $(MP)_p$ is well-posed for $p \in (1, 2 + \varepsilon_\Omega)$.
- In the setting where the sector is a creased domain, *i.e.* $\theta \in (0, \pi)$, Theorem 4 is in line with the earlier theory, and it makes explicit the dependence of ε_Ω on Ω (via the aperture θ).

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- For this range of θ's, let

$$p_{\text{critic}}(\theta) := \begin{cases} \min\left\{\frac{2\theta}{2\theta-\pi}, \frac{\theta}{\theta-\pi}\right\} & \text{if } \theta \in (\pi, 3\pi/2] \\ \min\left\{\frac{2\theta}{2\theta-\pi}, \frac{2\theta}{2\theta-3\pi}, \frac{\theta}{\theta-\pi}\right\} & \text{if } \theta \in (3\pi/2, 2\pi). \end{cases}$$

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- Then $\frac{4}{3} < p_{\text{critic}}(\theta) < 2$, so the range of indices for which $(MP)_p$ is well-posed is more restrictive than $(1, 2]$. This portion of the work should be compared with results of Brown, Capogna, and Lanzani who showed that in this scenario, $(MP)_p$ is solvable for some $p > 1$.

Thank you!