Higher regularity of the free boundary in the parabolic Signorini problem

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Joint work with Agnid Banerjee & Andrew Zeller

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Classical obstacle problem

We are given:

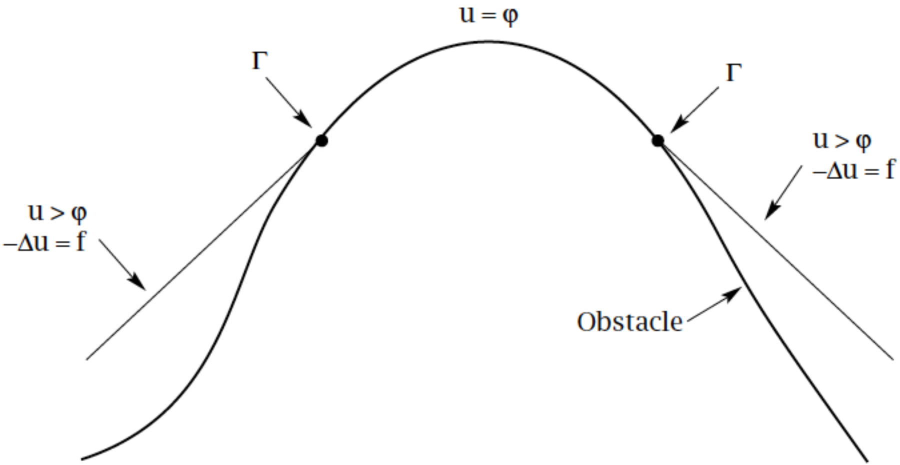
- $\phi \in C^2(D)$, the *obstacle*;
- $\psi \in W^{1,2}(D)$ with $\phi \leq \psi$ on ∂D, the *boundary values*;
- $f \in L^\infty(D)$, the *source term*.

We want to minimize

$$\int_D (|\nabla u|^2 + 2fu)dx$$

over $\mathcal{K} = \{u \in W^{1,2}(D) \ : u = \psi \text{ on } \partial D, u \geq \phi \text{ a.e. in } D\}$.

Classical obstacle problem



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First fundamental question: How smooth is the solution? The optimal regularity of the solution is $u \in C^{1,1}_{\text{loc}}(D) \cong W^{2,\infty}_{\text{loc}}(D)$.

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Second fundamental question: How smooth is the free boundary? In 1977 Kinderlherer and Nirenberg proved that, if the free boundary is a C^1 hypersurface, then it is C^ω (real analytic). Around the same time Caffarelli developed his theory of the regularity of the free boundary and proved Lipschitz regularity, and then proved how to go from Lipschitz to $C^{1,\alpha}$, using boundary Harnack principle.

A remark on higher regularity

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Used the Hodograph transform to prove that C^1 free boundaries for the classical obstacle problem are real analytic.

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Notice: Schauder estimates + Hopf Lemma $\Rightarrow \frac{v}{u}$ is $C^{k-1,\alpha}$ up the boundary. So De Silva-Savin gives regularity of the quotient one order higher than one might expect. Their result implies C^∞ regularity of $C^{1,\alpha}$ free boundaries, when $\varphi = 0$, for the classical obstacle problem.

The thin obstacle problem

We are given:

- $D \subset \mathbb{R}^n$: bounded domain;
- $\mathcal{M} \subset \partial D$: codimension one manifold,
- $\varphi : \mathcal{M} \to \mathbb{R}$, the *obstacle*;
- $\psi : \partial D \to \mathbb{R}$;

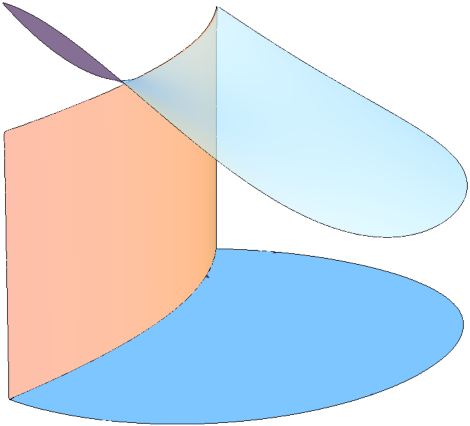
We want to minimize

$$\int_D |\nabla u|^2 dx, \tag{0.1}$$

over the convex set

$$\mathcal{K} = \{u \in W^{1,2}(D) \mid u = \psi \text{ on } \partial D \setminus \mathcal{M}, u \geq \varphi \text{ on } \mathcal{M}\}.$$

The thin obstacle problem



Where does the thin obstacle problem appear?

- In elasticity (Signorini), when an elastic body is at rest, partially laying on a surface $\mathcal{M}$.
- It models the flow of a saline concentration through a semipermeable membrane (osmosis, parabolic Signorini problem).
- In mathematical finance, when the random variation of an underlying asset changes discontinuously.

Our work

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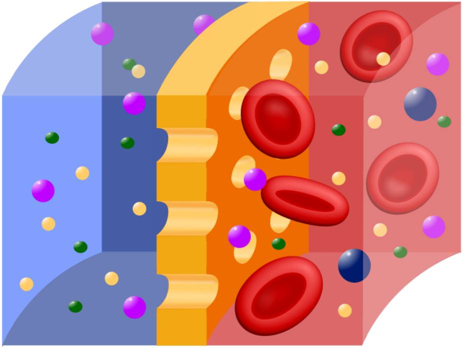
Other path: Prove a parabolic counterpart of De Silva-Savin, suited to our setting.

Semipermeable Membranes and Osmosis



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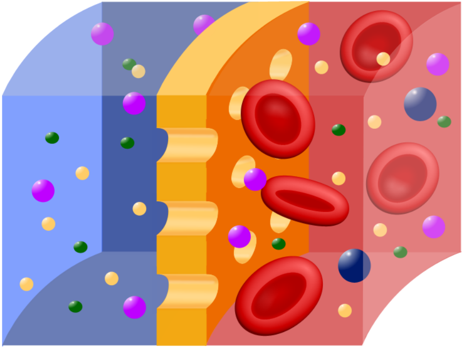

- Semipermeable membrane is a membrane that is permeable only for a certain type of molecules (solvents) and blocks other molecules (solutes).

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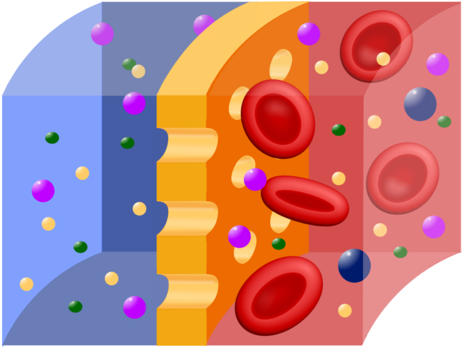

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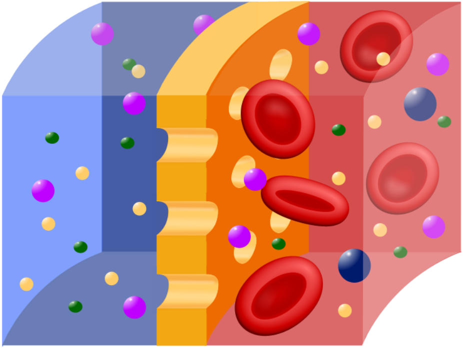

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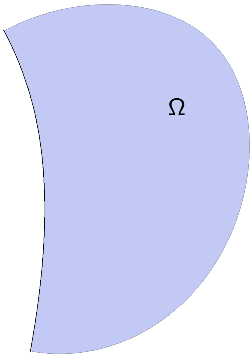
- The solvent flows through the membrane from the region of smaller concentration of solute to the region of higher concentration (osmotic pressure).
- The flow occurs in one direction. The flow continues until a sufficient pressure builds up on the other side of the membrane (to compensate for osmotic pressure), which then shuts the flow. This process is known as osmosis.

Mathematical Formulation

- We are given an open set $\Omega \subset \mathbb{R}^n$ and $\mathcal{M} \subset \partial\Omega$ semipermeable part of the boundary (thin manifold)



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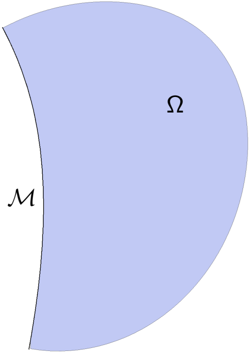
- We are given an open set $\Omega \subset \mathbb{R}^n$ and $\mathcal{M} \subset \partial\Omega$ semipermeable part of the boundary (thin manifold)
- $\phi : \mathcal{M}_T := \mathcal{M} \times (0, T] \to \mathbb{R}$ osmotic pressure (obstacle)



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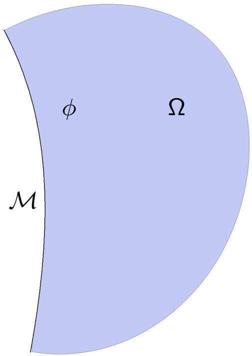
- $v : \Omega_T := \Omega \times (0, T] \to \mathbb{R}$ the pressure of the chemical solution, satisfies $\Delta v - \partial_t v = 0 \quad \text{in } \Omega_T$



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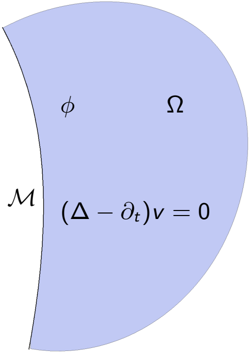

- On $\mathcal{M}_T$ we have the following boundary conditions

$$
\begin{aligned}
v > \phi \quad &\Rightarrow \quad \partial_\nu v = 0 && \text{(no flow)} \\
v \leq \phi \quad &\Rightarrow \quad \partial_\nu v = \lambda(v - \phi) && \text{(flow)}
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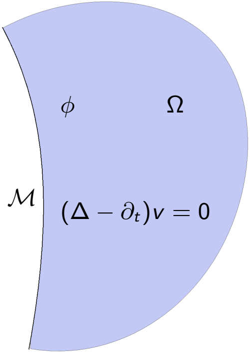

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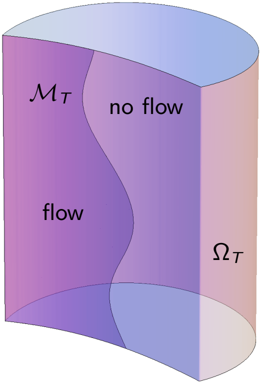

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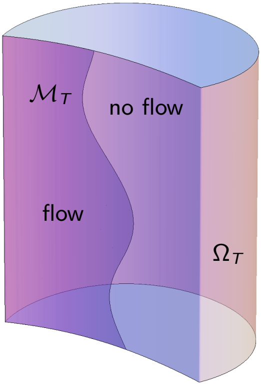
Parabolic Signorini Problem

- Letting $\lambda \to \infty$ we obtain the following conditions on $\mathcal{M}_T$ (*infinite permeability*)

$$v \geq \phi$$
$$\partial_\nu v \geq 0$$
$$(v - \phi)\partial_\nu v = 0$$



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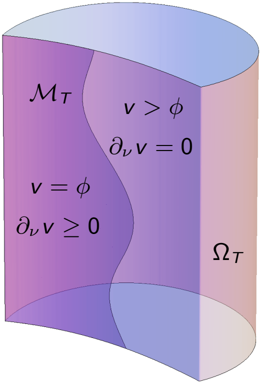

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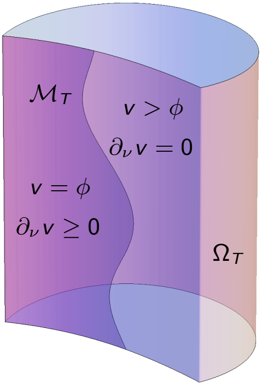
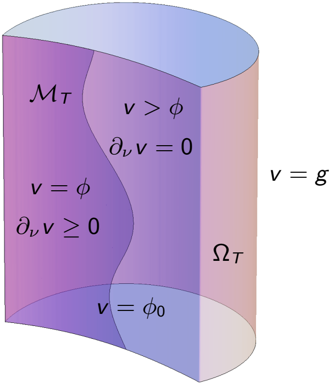
- Since v should stay above ϕ on $\mathcal{M}_T$, ϕ is known as the thin obstacle.



- The function $v(x,t)$ solves the following variational inequality:

$$\int_{\Omega_T} \nabla v \cdot \nabla (v-w) + \partial_t v (v-w) \geq 0$$

for all $w \in \mathcal{K}$



where
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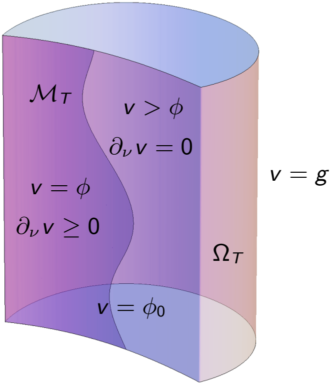

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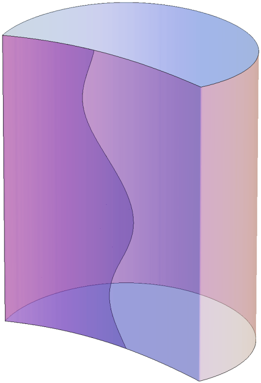
- Then, for any (reasonable) initial condition

$$v = \phi_0 \quad \text{on } \Omega_0 = \Omega \times \{0\}$$

the solution exists and is unique.

Free Boundary Problem

- The parabolic Signorini problem is another example of free boundary problem.

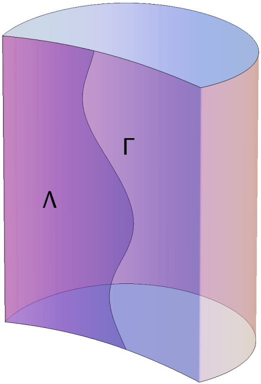


Free Boundary Problem

- The parabolic Signorini problem is another example of free boundary problem.
- Let $\Lambda_\phi(v) := \{(x,t) \in \mathcal{M}_T : v = \phi\}$ be coincidence set. Then,

$$\Gamma_\phi(v) := \partial_{\mathcal{M}_T} \Lambda_\phi(v)$$

is the free boundary.



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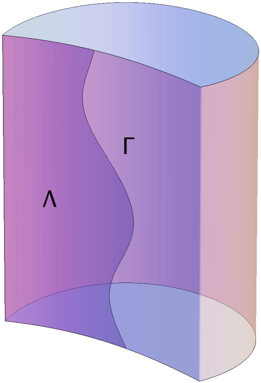
  is the free boundary.
- One is interested in the structure, geometric properties and the regularity of the free boundary.



The parabolic Signorini problem

Let $\Omega \subset \mathbb{R}^n, \mathcal{M}$ be a relatively open subset of $\partial\Omega$, $\mathcal{S} = \partial\Omega \setminus \mathcal{M}$.

$$\begin{aligned} \Delta v - \partial_t v &= 0 && \text{in } \Omega_T := \Omega \times [0, T], \\ v \geq \phi, \quad \partial_\nu v \geq 0, \quad (v-\phi)\partial_\nu v &= 0 && \text{on } \mathcal{M}_T := \mathcal{M} \times (0, T], \\ v &= g && \text{on } \mathcal{S}_T := \mathcal{S} \times (0, T], \\ v(\cdot, 0) &= \phi_0 && \text{on } \Omega_0 := \Omega \times \{0\}, \end{aligned}$$

where ∂_ν is the outer normal derivative on $\partial\Omega$ and $\phi : \mathcal{M}_T \to \mathbb{R}$, $\phi_0 : \Omega_0 \to \mathbb{R}$ and $g : \mathcal{S}_T \to \mathbb{R}$ are given.

Optimal regularity of the solution:

Danielli, Garofalo, Petrosyan & To, 2013:

$v \in H^{3/2,3/4}_{\text{loc}}(\Omega_T \cup \mathcal{M}_T)$.

The parabolic Signorini problem

$$\text{Free boundary} = \Gamma = \partial_{\mathcal{M}_T}\{(x,t) \in \mathcal{M}_T \mid v(x,t) > \phi(x,t)\}.$$

A classification of free boundary points is achieved by proving the monotonicity of a generalization of Almgren's frequency function.

This is a function of r, where r denotes the radius of balls centered around a fixed free boundary point.

Almgren monotonicity formula

Historical background: Almgren's monotonicity formula

Crucial tool: fundamental monotonicity formula proved in 1979 by F. Almgren: if $\Delta u = 0$ in B_1, then the frequency of u, given by

$$r \to N(u,r) = \frac{r \int_{B_r} |\nabla u|^2}{\int_{S_r} u^2},$$

is increasing in $(0,1)$.

Since the generalization of Almgren's frequency for the parabolic Signorini problem is a bounded, monotone non-decreasing function, it has a limit as $r \to 0$. This limit was proved to be $3/2$ or ≥ 2.

This leads to the definition of regular points of the free boundary, which are points on the free boundary where the limit of the frequency function, as $r \to 0$, is $3/2$.

Our main result: the free boundary near regular points of the parabolic thin obstacle problem with zero obstacle is C^∞ regular in space and time.

Initial regularity of the regular set

Recall: regular points of the free boundary are the points where the frequency function tends to 3/2 as $r \to 0$.

Danielli, Garofalo, Petrosyan & To, 2013:

If 0 is a regular free boundary point, $\exists \delta, \alpha > 0$ and g with $\nabla_{x''} g \in H^{\alpha,\alpha/2}(B''_\delta \times (-\delta^2, 0])$, where $B''_r := B_r \cap \mathbb{R}^{n-2}$, such that

$$\Gamma \cap \left(B'_\delta \times (-\delta^2, 0]\right) = \{(x', t) \in B'_\delta \times (-\delta^2, 0] \mid x_{n-1} = g(x'', t)\}.$$

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Petrosyan-Zeller, 2015: v_t is Hölder continuous at regular free boundary points.
Applying boundary Harnack principle to $\frac{v_t}{v_{x_{n-1}}}$, one concludes g_t is Hölder continuous.
Hence the regular set is locally a $C^{1,\alpha}$ hypersurface in x' and t.
What about higher regularity?

Our main result

Banerjee, SVG, Zeller, 2017:

Assuming $\varphi \equiv 0$, The free boundary near regular points of the parabolic thin obstacle problem with zero obstacle is C^∞ regular in space and time, that is, g is locally C^∞.

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Assuming $\varphi \equiv 0$, The free boundary near regular points of the parabolic thin obstacle problem with zero obstacle is C^∞ regular in space and time, that is, g is locally C^∞.

Main ingredient of the proof: parabolic counterpart of De Silva-Savin for "slit" domains.

De Silva-Savin, 2014:

[Higher order boundary Harnack principle] Assume $D \subset \mathbb{R}^n$ is a $C^{k,\alpha}$ domain, $0 \in \partial D$. Let u, v be harmonic functions vanishing on $\partial D \cap B_1$. Assume $u > 0$ in D. Then

$$\left|\left|\frac{v}{u}\right|\right|_{C^{k,\alpha}(B_{1/2})} \leq C||v||_{L^\infty(B_1)}.$$

Banerjee, SVG, Zeller: parabolic counterpart to the higher order boundary Harnack for "slit" domains

Notations

Let $g \in H^{k+1+\alpha}$.
We define a slit as

$$\mathcal{P} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid x_n = 0, x_{n-1} \leq g(x'', t)\}.$$

Define

$\Gamma = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid x_n = 0, x_{n-1} = g(x'', t)\},$

$\Psi_r = \{(x', x_n, t) \mid -r^2 < t \leq 0, |x_n| < 2r, |x'| < r\} \subset \mathbb{R}^n \times \mathbb{R}.$

We will look at $\Psi_1 \setminus \mathcal{P}$.

Main result

Banerjee, SVG, Zeller, 2017:

[Higher order boundary Harnack principle] Let $k \geq 0$. Assume $U > 0$ solves $\Delta U - U_t = 0$ and u solves $\Delta u - u_t = \frac{U_0}{r} f$ in $\Psi_1 \setminus \mathcal{P}$, where U_0 and r are specific functions and f has an appropriate regularity assumption. Assume $U, u \in C(\Psi_1)$, and that U and u vanish continuously on $\mathcal{P}$, where

$$\mathcal{P} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid x_n = 0, x_{n-1} \leq g(x'', t)\}$$

and $g \in H^{k+1+\alpha}$. If $k = 0$, we assume $||g||_{C^{1+\alpha}} \leq 1$. Then

$$\left|\left|\frac{u}{U}\right|\right|_{H^{k+1+\alpha}(\Gamma \cap \Psi_{1/2})} \leq C.$$

How do we use this to prove higher regularity?

Recall: $\Gamma \cap \left(B'_\delta \times (-\delta^2, 0]\right) = \{(x', t) \in B'_\delta \times (-\delta^2, 0] \mid x_{n-1} = g(x'', t)\}$.
We have $v(x'', g(x'', t), 0, t) = 0$. Differentiate w.r.t. $x_1, \ldots, x_{n-2}, t$:

$$\frac{D_i v}{D_{n-1} v} = -D_i g, \ \frac{D_t v}{D_{n-1} v} = -D_t g.$$

In our higher order boundary Harnack principle, let $u = D_i v$, $U = D_{n-1} v$.
Then $D_i g \in H^{1+\alpha}$.

In our higher order boundary Harnack principle, let $u = D_t v$, $U = D_{n-1} v$.
Then $D_t g \in H^{1+\alpha}$.

Hence $g \in H^{2+\alpha}$.

Proceed inductively.

Thank you!