Two-phase free boundary problems for harmonic measure with Hölder data (and blowups in multi-phase problems)

Joint work with
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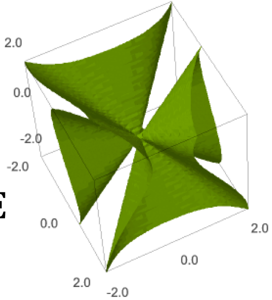

AMS Meeting Boston

Special Session on Regularity of PDE

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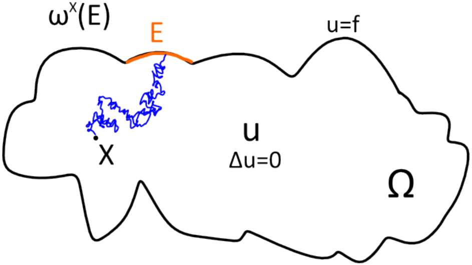
Dirichlet Problem and Harmonic Measure

Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a regular domain for (D).



Dirichlet Problem

Given $f \in C_c(\partial\Omega)$,
find $u \in C^2(\Omega) \cap C(\overline{\Omega})$:

$$(D) \begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = f \text{ on } \partial\Omega \end{cases}$$

$$\Delta = \partial_{x_1 x_1} + \partial_{x_2 x_2} + \cdots + \partial_{x_n x_n}$$

$\exists!$ family of probability measures $\{\omega^X\}_{X \in \Omega}$ on the boundary $\partial\Omega$ called **harmonic measure** of Ω with pole at $X \in \Omega$ such that

$$u(X) = \int_{\partial\Omega} f(Q) d\omega^X(Q) \qquad \text{solves (D)}$$

For unbounded domains, we may also consider harmonic measure with pole at infinity.

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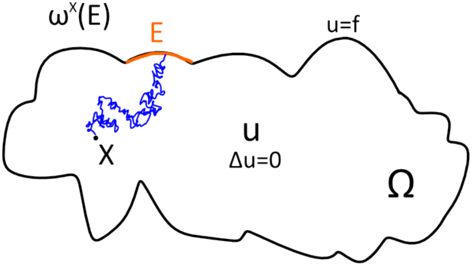

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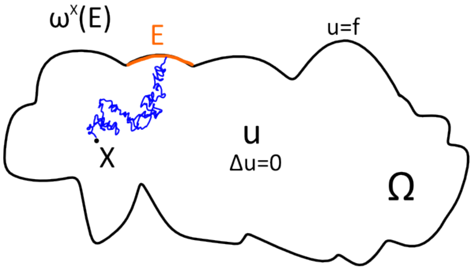

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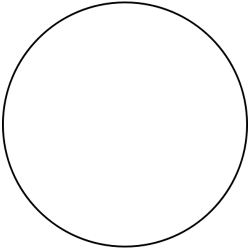
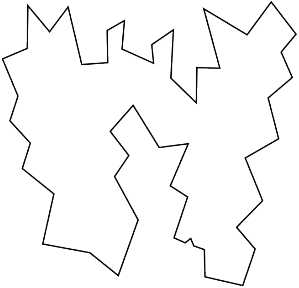
Examples of Regular Domains

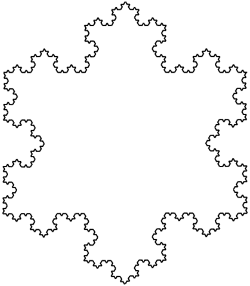
NTA domains introduced by Jerison and Kenig 1982:
Quantitative Openness + Quantitative Path Connectedness



Smooth Domains



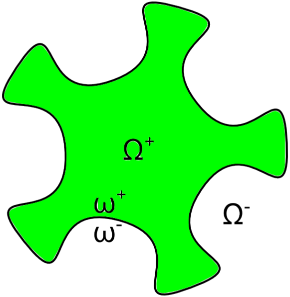
Lipschitz Domains



Quasispheres

(e.g. snowflake)

Two-Phase Free Boundary Regularity Problem



$\Omega \subset \mathbb{R}^n$ is a **2-sided domain** if:

1. $\Omega^+ = \Omega$ is open and connected
2. $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$ is open and connected
3. $\partial\Omega^+ = \partial\Omega^-$

Let $\Omega \subset \mathbb{R}^n$ be a 2-sided domain, equipped with interior harmonic measure ω^+ and exterior harmonic measure ω^-.

If $\omega^+ \ll \omega^- \ll \omega^+$, then $f = \dfrac{d\omega^-}{d\omega^+}$ exists, $f \in L^1(d\omega^+)$.

Determine the extent to which existence or regularity of f controls the geometry or regularity of the boundary $\partial\Omega$.

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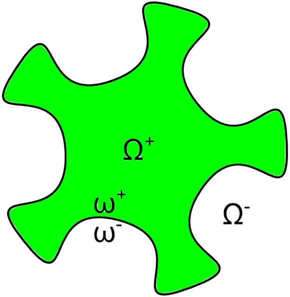

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Regularity of a boundary can be expressed in terms of geometric blowups of the boundary

Existence of Measure-Theoretic Tangents at Typical Points

Theorem (Azzam-Mourgoglou-Tolsa-Volberg 2016)

Let $\Omega \subset \mathbb{R}^n$ be a 2-sided domain equipped with harmonic measures $\omega^\pm$ on $\Omega^\pm$. If $\omega^+ \ll \omega^- \ll \omega^+$, then $\partial\Omega = G \cup N$, where

1. $\omega^\pm(N) = 0$ *and* $\mathcal{H}^{n-1} \llcorner G$ *is locally finite,*
2. $\omega^\pm \llcorner G \ll \mathcal{H}^{n-1} \llcorner G \ll \omega^\pm \llcorner G$,
3. *up to a* $\omega^\pm$*-null set,* G *is contained in a countably union of graphs of Lipschitz functions* $f_i : V_i \to V_i^\perp$, $V \in G(n, n-1)$.

In contemporary Geometric Measure Theory, we express (3) by saying $\omega^\pm$ are $(n-1)$-dimensional **Lipschitz graph rectifiable**.

In particular, if $\omega^+ \ll \omega^- \ll \omega^+$, then at $\omega^\pm$-a.e. $x \in \partial\Omega$, there is a unique $\omega^\pm$-approximate tangent plane $V \in G(n, n-1)$:

$$\limsup_{r\downarrow 0} \frac{\omega^\pm(B(x,r))}{r^{n-1}} > 0 \quad \text{and} \quad \limsup_{r\downarrow 0} \frac{\omega^\pm(B(x,r) \setminus \mathrm{Cone}(x+V,\alpha))}{r^{n-1}} = 0$$

for every cone around the $(n-1)$-plane $x + V^\perp$.

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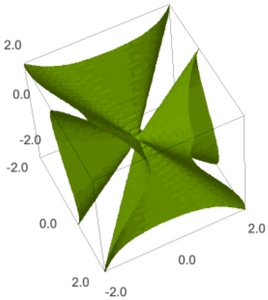

Figure: The zero set of Szulkin's degree 3 harmonic polynomial $p(x, y, z) = x^3 - 3xy^2 + z^3 - 1.5(x^2 + y^2)z$

$\Omega^{\pm} = \{p^{\pm} > 0\}$ is a 2-sided domain, $\omega^+ = \omega^-$ (pole at infinity), $\log \frac{d\omega^-}{d\omega^+} \equiv 0$ but $\partial\Omega^{\pm} = \{p = 0\}$ is not smooth at the origin.

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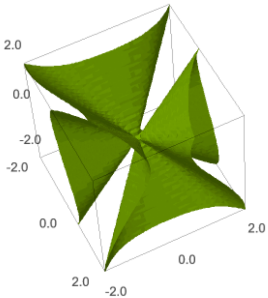

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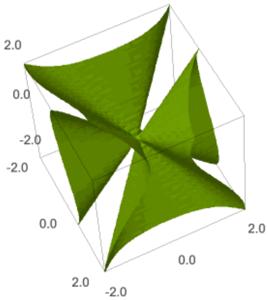

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Useful Terminology: Local Set Approximation (B-Lewis)

Let $A \subset \mathbb{R}^n$ be closed, let $x_i \in A$, let $x_i \to x \in A$, and let $r_i \downarrow 0$.

If $\dfrac{A - x}{r_i} \to T$, we say that T is a **tangent set** of A at x.

- Attouch-Wets topology: $\Sigma_i \to \Sigma$ if and only if for every $r > 0$, $\lim_{i\to\infty} \left(\sup_{x\in\Sigma_i\cap B_r} \operatorname{dist}(x, \Sigma) + \sup_{y\in\Sigma\cap B_r} \operatorname{dist}(y, \Sigma_i)\right) = 0$
- There is at least one tangent set at each $x \in A$.
- There could be more than one tangent set at each $x \in A$.

If $\dfrac{A - x_i}{r_i} \to S$, we say that S is a **pseudotangent set** of A at x.

- Every tangent set of A at x is a pseudotangent set of A at x.
- There could be pseudotangent sets that are not tangent sets.

We say that A is **locally bilaterally well approximated by $\mathcal{S}$** if every pseudotangent set of A belongs to $\mathcal{S}$.

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Tangents and Pseudotangents under Weak Regularity

Theorem (Kenig-Toro 2006, B 2011, B-Engelstein-Toro 2017)

Let $\Omega \subset \mathbb{R}^n$ be a 2-sided domain equipped with harmonic measures $\omega^\pm$ on $\Omega^\pm$. If $\Omega^\pm$ are NTA and $f = \dfrac{d\omega^-}{d\omega^+}$ has $\log f \in \mathrm{VMO}(d\omega^+)$, then

- *$\partial\Omega$ is locally bilaterally well approximated by zero sets of harmonic polynomials $p : \mathbb{R}^n \to \mathbb{R}$ of degree at most d_0 such that $\Omega_p^\pm = \{x : \pm p(x) > 0\}$ are NTA domains and $\dim_M \partial\Omega = n - 1$.*

Moreover, we can partition $\partial\Omega = \Gamma_1 \cup S = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_{d_0}$.

- *Γ_1 is relatively open in $\partial\Omega$, Γ_1 is locally bilaterally well approximated by $(n-1)$-planes, and $\dim_M \Gamma_1 = n - 1$*
- *S is closed, $\omega^\pm(S) = 0$, and $\dim_M S \le n - 3$*
- *$S = \Gamma_2 \cup \cdots \cup \Gamma_{d_0}$, where $x \in \Gamma_d \Leftrightarrow$ every tangent set of $\partial\Omega$ at x is the zero set of a homogeneous harmonic polynomial q of degree d such that $\Omega_q^\pm$ are NTA domains.*

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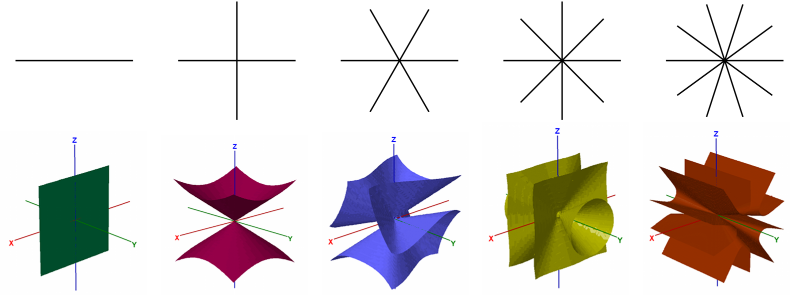
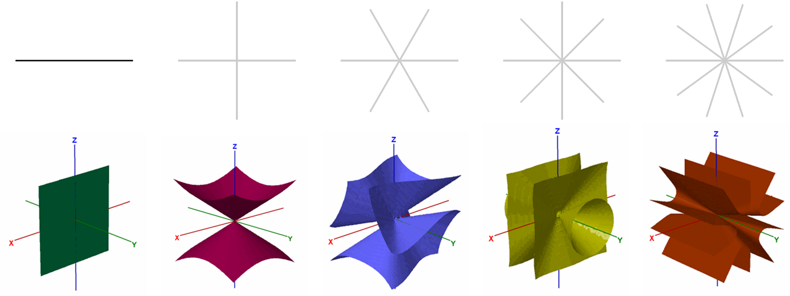


Zero Sets of HHP in $\mathbb{R}^2$ and $\mathbb{R}^3$ of Degrees 1, 2, 3, 4, 5

Admissible Tangents

- In first row, only the first example (degree 1) separates the plane into 2-sided NTA domains
- In second row, only the first, third, and fifth examples (odd degrees) separate space into 2-sided NTA domains
- In $\mathbb{R}^4$ or higher dimensions, there are examples of all degrees that separate space into 2-sided NTA domains



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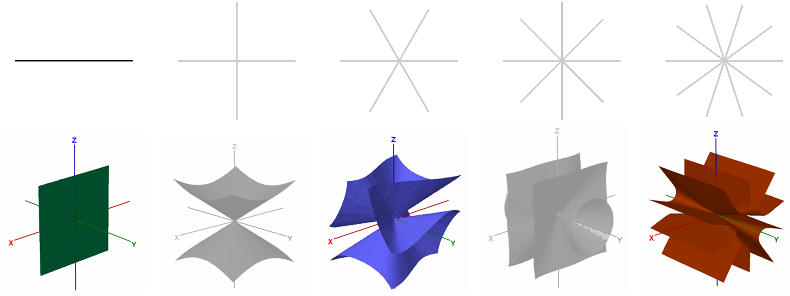

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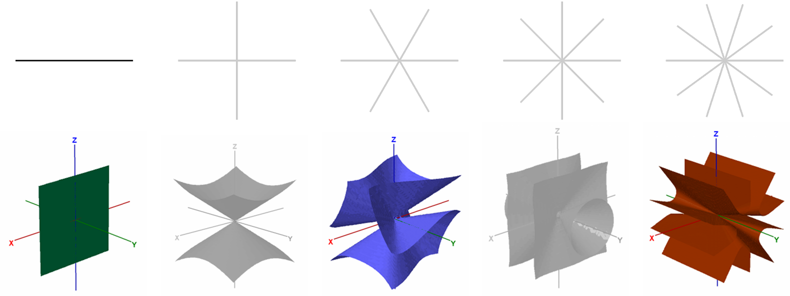

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Regularity under Hölder and Higher Order Data

2-sided NTA + $\log \frac{d\omega^-}{d\omega^+} \in \mathrm{VMO}(d\omega^+) \Longrightarrow \partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_{d_0}$

Theorem (Engelstein 2016)

- *Hölder regularity: If* $\log \frac{d\omega^-}{d\omega^+} \in C^{0,\alpha}$*, then* Γ_1 *is* $C^{1,\alpha}$*.*
- *Higher regularity: If* $\log \frac{d\omega^-}{d\omega^+} \in C^{\infty}$*, then* Γ_1 *is* C^{∞}*.*

Theorem (B-Engelstein-Toro 2018)

Assume that $\log \frac{d\omega^-}{d\omega^+} \in C^{0,\alpha}$*. Then:*

- *At every* $x \in \partial\Omega$*, there is a unique tangent set of* $\partial\Omega$ *at* x*.*
- *The singular set* $S = \Gamma_2 \cup \cdots \cup \Gamma_{d_0}$ *is* $C^{1,\beta}$ $(n-3)$*-rectifiable:* S *is subset of a countable union of* $C^{1,\beta}$ *submanifolds* M_i^{n-3}

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Assume that $\log \frac{d\omega^-}{d\omega^+} \in C^{0,\alpha}$*. Then:*

- *At every* $x \in \partial\Omega$*, there is a unique tangent set of* $\partial\Omega$ *at* x*.*
- *The singular set* $S = \Gamma_2 \cup \cdots \cup \Gamma_{d_0}$ *is* $C^{1,\beta}$ *$(n-3)$-rectifiable:* S *is subset of a countable union of* $C^{1,\beta}$ *submanifolds* M_i^{n-3}

Remarks / Ingredients in the Proof

2-sided NTA + $\log \frac{d\omega^-}{d\omega^+} \in \mathrm{VMO}(d\omega^+) \Longrightarrow \partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_{d_0}$

Theorem (B-Engelstein-Toro 2017) $\dim_M(\Gamma_2 \cup \cdots \cup \Gamma_{d_0}) \leq n-3$

- Do not have monotonicity nor a definite rate of convergence of $(\partial\Omega - x_i)/r_i$ to Σ_p. Do not know that tangents of $\partial\Omega$ are unique.
- Instead: we use Local Set Approximation framework (B-Lewis) + prove "excess improvement" type lemma for pseudotangents
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- For each $x \in \Gamma_d$, establish almost monotonicity of a Weiss-type functional $W_d(r, x; v^x)$, where $v^x(z) = \frac{d\omega^-}{d\omega^+}(x)u^+(z) - u^-(z)$ and $u^\pm$ are Green's functions associated to $\Omega^\pm$
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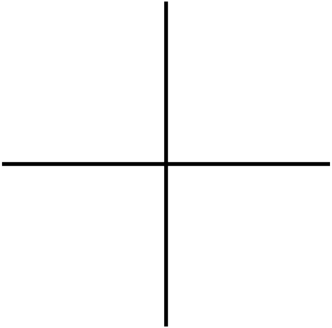
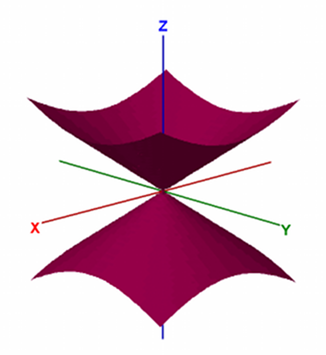
What about the missing harmonic polynomials?





Multiphase Free Boundary Regularity Problem

An **NTA configuration** $\mathbf{\Omega} = (\{\Omega_i\}, \Sigma)$ is a partition of $\mathbb{R}^n$ into finitely many NTA domains Ω_i (the "chambers") and a closed set Σ (the "interface") such that

$$\mathbb{R}^n = \Sigma \cup \bigcup_i \Omega_i, \qquad \Sigma = \bigcup_i \partial\Omega_i.$$

The **valency** of $x \in \Sigma$ is the number of chambers with $x \in \partial\Omega_i$.

Multiphase Problem (Akman-B):

Let ω_i denote harmonic measure on the chamber Ω_i of $\mathbf{\Omega}$.

If $\omega_i \ll \omega_j \ll \omega_i$ on $\partial\Omega_i \cap \partial\Omega_j$, then $f_j^i = \dfrac{d\omega_i}{d\omega_j} \in L^1(d\omega_j)$.

Determine the extent to which simultaneous existence or regularity of the f_j^i along $\partial\Omega_i \cap \partial\Omega_j$ controls the geometry or regularity of the interface Σ.

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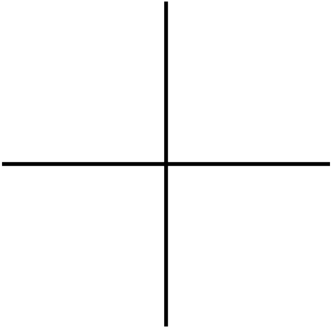
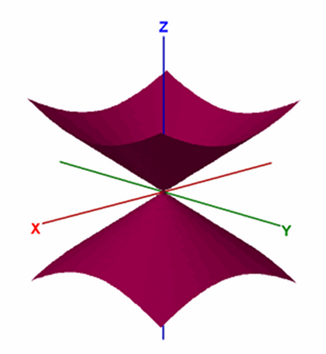
Example: $\omega_i = \omega_j$ on $\partial\Omega_i \cap \partial\Omega_j$ (pole at infinity)





Sample Result: Blowups at Bipartite Points

Let $\mathbf{\Omega} = (\{\Omega_i\}, \Sigma)$ be an NTA configuration in $\mathbb{R}^n$ and let $x \in \Sigma$.
The **two-phase graph** (V, E) of $\mathbf{\Omega}$ at x is defined so that

- The vertices of the graph are the chambers Ω_i with $x \in \partial\Omega_i$
- Two chambers $\Omega_i, \Omega_j \in V$ are connected by an edge if and only if $\Omega_i \neq \Omega_j$ and there exists $y \in \partial\Omega_i \cap \partial\Omega_j$ with valency 2.

We say $x \in \Sigma$ is **bipartite** if the two-phase graph of $\mathbf{\Omega}$ at x is bipartite.

Theorem (Akman-B 2018)

Let $\mathbf{\Omega} = (\{\Omega_i\}, \Sigma)$ be an NTA configuration in $\mathbb{R}^n$. Assume that

$$\log \frac{d\omega_i}{d\omega_j} \in VMO(d\omega_j|_{\partial\Omega_i \cap \partial\Omega_j}) \quad \text{for all } i, j.$$

If $x \in \Sigma$ is bipartite, then there is $d = d(x)$ such that every tangent set of Σ at x is the zero set Σ_q of a hhp q of degree d.

- *Σ_q determines an NTA configuration $\mathbf{\Omega}_q$*
- *the two-phase graph of $\mathbf{\Omega}_q$ at 0 is isomorphic to the two-phase graph of $\mathbf{\Omega}$ at x.*

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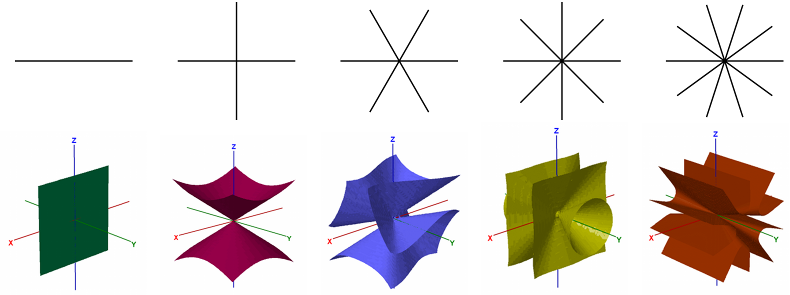


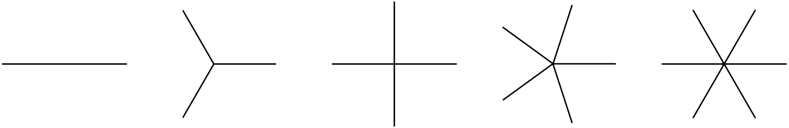
Zero Sets of HHP in $\mathbb{R}^2$ and $\mathbb{R}^3$ of Degrees 1, 2, 3, 4, 5

Every point in the zero set of a non-constant harmonic function is bipartite by the mean value property

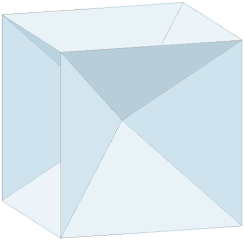
All homogeneous harmonic polynomials whose zero sets determine an NTA configuration occur as tangents in the Multiphase Problem

Further Work In Progress (Akman-B)

We expect to classify all tangent sets of the interfaces of **planar NTA configurations** with VMO free boundary conditions:



Example (A Platonic Cone): The NTA configuration whose interface is the cone over the skeleton of the cube in $\mathbb{R}^3$ has equal harmonic measures with pole at infinity on all 6 chambers, but the origin is not bipartite.



What are all of the possible tangent sets in $\mathbb{R}^3$?

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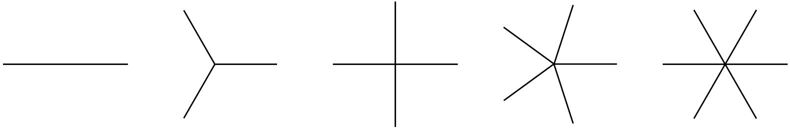

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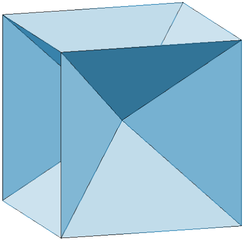

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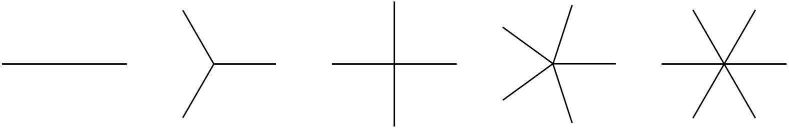

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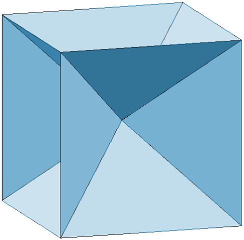

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Thank You!