Sufficient conditions for $C^{1,\alpha}$ parametrization

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April 22, 2018

AMS Spring Eastern Sectional Meeting
Special Session on Regularity of PDEs on Rough Domains

- Reifenberg 1960: a “flat” set can be parametrized by a Hölder map.
  - The set is required to be flat and without holes: at every point and scale there’s a plane close to the set and the set is close to the plane (official definition coming soon)

- David-Kenig-Toro 2001: a “flat” set with small β numbers can be parametrized by a $C^{1,\alpha}$ map
  - The sets are “flat” with *vanishing constant*
- Kolasiński 2015: a “flat” set with small holes and small β numbers can be parametrized by a $C^{1,\alpha}$ map
  - Small holes = size of β
  - Uses Menger-like curvatures

- David-Toro 2012: a “flat” set with holes can be parametrized by a Hölder map
  - Moreover if we assume convergence of a Jones function then we can get a bi-Lipschitz parametrization
  - No control assumed on the size of the holes

- G. 2017: a “flat” set with holes can be parametrized by a $C^{1,\alpha}$ map if we assume a stronger convergence of the Jones function
  - Again, no control assumed on the size of the holes

Definition

Let $E \subseteq \mathbb{R}^n$ and let $\varepsilon > 0$. Define E to be *Reifenberg flat* if the following conditions (1) hold.

(1) For $x \in E$, $0 < r \leq 10$ there is a d-plane $P(x,r)$ such that

$$\begin{aligned} \operatorname{dist}(y, P(x,r)) &\leq \varepsilon, & y &\in E \cap B(x,r), \\ \operatorname{dist}(y, E) &\leq \varepsilon, & y &\in P(x,r) \cap B(x,r). \end{aligned}$$

Definition

Let $E \subseteq \mathbb{R}^n$ and let $\varepsilon > 0$. Define E to be *Reifenberg flat* with holes if the following conditions (1)-(2) hold.

(1) For $x \in E$, $0 < r \leq 10$ there is a d-plane $P(x, r)$ such that

$$\mathrm{dist}(y, P(x,r)) \leq \varepsilon, \qquad y \in E \cap B(x,r),$$
$$\sout{\mathrm{dist}(y, E) \leq \varepsilon, \qquad y \in P(x,r) \cap B(x,r).}$$

(2) Moreover we require some compatibility between the $P(x, r)$'s:

$$d_{x,10^{-k}}(P(x,10^{-k}), P(x,10^{-k+1})) \leq \varepsilon, x \in E,$$
$$d_{x,10^{-k+2}}(P(x,10^{-k}), P(y,10^{-k})) \leq \varepsilon, x, y \in E, \ |x-y| \leq 10^{-k+2}$$

Let $E \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$, and $r > 0$.

Definition

$$\beta_\infty^E(x,r) = \inf_P \sup_{y \in E \cap B(x,r)} \frac{\mathrm{dist}(y,P)}{r}$$

if $E \cap B(x,r) \neq \varnothing$, where the infimum is taken over all d-planes P, and $\beta_\infty^E(x,r) = 0$ if $E \cap B(x,r) = \varnothing$.

Definition

$$\beta_p^E(x,r) = \inf_P \left\{ \int_{E \cap B(x,r)} \left(\frac{\mathrm{dist}(y,P)}{r} \right)^p \frac{d\mathcal{H}^d(y)}{r^d} \right\}^{\frac{1}{p}}$$

where the infimum is taken over all d-planes P.

Theorem (G. David, T. Toro, 2012)

Let $E \subseteq \mathbb{R}^n$ be a Reifenberg flat set with holes. Then we can construct a map $f : \mathbb{R}^d \to \mathbb{R}^n$, such that $E \subset f(\mathbb{R}^d)$ and f is bi-Hölder. Moreover, if we assume that there exists $M < +\infty$ such that

$$\sum_{k \geq 0} \beta_\infty^E(x, r_k)^2 \leq M, \qquad \text{for all } x \in E,$$

then f is bi-Lipschitz.

Theorem (G. David, T. Toro, 2012)

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$$\sum_{k \geq 0} \beta_1^E(x, r_k)^2 \leq M, \qquad \text{for all } x \in E,$$

then f is bi-Lipschitz.

Theorem (G., 2017)

Let $E \subseteq \mathbb{R}^n$ be a Reifenberg flat set with holes and $\alpha \in [0,1]$. Also assume that there exists $M < +\infty$ such that

$$\sum_{k \geq 0} \frac{\beta_\infty^E(x, r_k)^2}{r_k^\alpha} \leq M, \qquad \text{for all } x \in E.$$

Then we can construct a map $f \colon \mathbb{R}^d \to \mathbb{R}^n$, such that $E \subset f(\mathbb{R}^d)$ such that the map and its inverse are $C^{1,\alpha}$ continuous. Moreover the Hölder constants depend only on n, d, and M.

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Why?

Why did I prove it?

- Characterize different types of rectifiability
- Connection between decay of β's and smoothness

Theorem (G., 2017)

Let μ be a Radon measure on $\mathbb{R}^n$ such that $0 < \theta^{d*}(\mu, x) < \infty$ for μ-a.e. x. Assume that for μ-a.e. $x \in \mathbb{R}^n$,

$$J^{\mu}_{2,\alpha}(x) = \sum_{k \geq 0} \frac{\beta^{\mu}_2(x, r_k)^2}{r_k^{\alpha}} < \infty.$$

Then μ is (countably) $C^{1,\alpha}$ d-rectifiable.

Theorem (G., 2017)

Let $E \subseteq \mathbb{R}^n$ such that $0 < \theta^{d*}(E, x) < \infty$, for a.e. $x \in E$. Assume that for almost every $x \in E$,

$$J^E_{\infty,\alpha}(x) = \sum_{k \geq 0} \frac{\beta^E_\infty(x, r_k)^2}{r_k^\alpha} < \infty.$$

Then E is (countably) $C^{1,\alpha}$ d-rectifiable.

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