On quantitative absolute continuity of harmonic measure and big piece approximation by chord-arc domains

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Introduction/History

- F. and M. Riesz (1916): $\Omega \subset \mathbb{C}$, simply connected. Then $\partial\Omega$ rectifiable implies $\omega \ll \sigma$.

- C.E. due to C. Bishop and P. Jones (1990): conclusion need not hold w/o some connectivity.

Notation: $\omega =$ harmonic measure (at generic point in Ω), $\sigma = \mathcal{H}^1\lfloor_{\partial\Omega}$ (or $\sigma = \mathcal{H}^{d-1}\lfloor_{\partial\Omega}$ in $\mathbb{R}^d$).

Recall: $\partial\Omega$ rectifiable $=$ covered by a countable union of Lipschitz graphs, up to a set of $\mathcal{H}^1$ (or $\mathcal{H}^{d-1}$) measure 0.

Introduction/History (continued)

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Remark: it follows that Dirichlet problem solvable with L^p data, some $p < \infty$ (in fact, in Lip domain can take $p = 2$ or even $2 - \varepsilon$).

Introduction/History (continued)

A_∞ **more precisely:**

- $\omega \in A_\infty(\sigma)$ means that $\forall B$ centered on $\partial\Omega$ with $r_B < \operatorname{diam}(\partial\Omega)$, and $\forall$ Borel $E \subset \Delta := B \cap \partial\Omega$, $X \in \Omega \setminus 4B$

$$\omega^X(E) \lesssim \left(\frac{\sigma(E)}{\sigma(\Delta)}\right)^\theta \omega^X(\Delta).$$

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- Note that A_∞ and weak-A_∞ are each quantitative, scale invariant versions of absolute continuity.

Intro/History (continued)

- David-Jerison (1990), and independently Semmes: Ω "chord-arc" domain (aka CAD) in $\mathbb{R}^{n+1}$, then $\omega \in A_\infty(\sigma)$.

Definition: CAD = NTA + ADR boundary

$$\mathrm{ADR}: \quad \sigma\big(\Delta(x,r)\big) \approx r^n$$

$$\mathrm{NTA} \;=\; \mathrm{int.\ and\ ext.\ Corkscrew\,(CS)} \;+\; \mathrm{Harnack\ Chains\,(HC)}$$

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- CS: $\exists B' \subset B \cap \Omega$, with $r_{B'} \approx r_B$; denote by X_B = center of B'; this is a "CS point relative to B".
- HC: quantitative scale invariant path connectedness.

Intro/History (continued)

Method of proof of [DJ]: ADR + 2-sided CS implies "Interior Big Pieces of Lipschitz Sub-Domains" (IBPLSD); i.e., for every B centered on $\partial\Omega$, with $r_B < \text{diam}(\partial\Omega)$, $\exists$ subdomain $\Omega_B \subset \Omega \cap B$ s.t.

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- Remark: $\exists$ a refinement of this result due to M. Badger in absence of upper ADR bound.

Intro/History (continued)

Q: why does this give A_∞?

- IBPLSD implies: by Dahlberg (applied in Ω_B), plus maximum principle, obtain $\exists \eta \in (0,1)$ s.t. for Borel $E \subset \Delta$,

$$(*) \qquad \sigma(E) \geq (1-\eta)\sigma(\Delta) \quad \Longrightarrow \quad \omega^{X_B}(E) \gtrsim 1\,.$$

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- Then use pole change formula for harmonic measure (uses HC), to change scales, i.e., to improve to $\omega \in A_\infty(\sigma)$.

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- Again by [DJ] have IBPLSD, hence again have (*).
- w/o HC, pole change formula unavailable; [BL] argument "changes pole w/o pole change formula", this (necessarily) introduces errors which result in non-doubling; weak-A_∞ is best possible conclusion.

Converses

Some Converse results:

- Lewis - Vogel (2007): $\partial\Omega$ ADR, $\omega \approx \sigma$; i.e., $k := \frac{d\omega}{d\sigma} \approx 1$ (after normalizing). Then $\partial\Omega$ is Uniformly Rectifiable (UR) (quantitative scale invariant version of rectifiability - David-Semmes).

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Proof idea (both papers), based on Alt-Caffarelli technique: small oscillation of ∇G plus non-degeneracy of ∇G implies flatness.

Recent Results (posted late 2017- early 2018)

- J. Azzam: $\partial\Omega$ ADR, then

$$\omega \in A_\infty(\sigma) \quad \Longleftrightarrow \quad \partial\Omega \text{ UR and } \Omega \text{ "semi-uniform" (S-U).}$$

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- $\omega \in A_\infty \implies \partial\Omega$ UR by S.H. - Martell.
- UR + S-U implies IBPCAD; so, get (*) by M.P. + [DJ], improve to weak-A_∞ by [BL], then S-U gives doubling, hence A_∞.

Recent Results (continued)

Remark: note that connectivity in Azzam's result (S-U condition) is about *doubling*, not about absolute continuity.

OTOH, in light of Bishop-Jones example, the question remains: what is minimal connectivity assumption, which, in conjunction with UR, yields quantitative absolute continuity of harmonic measure?

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- Combining work of two different groups of authors, we can now answer this.

Recent Results (continued)

Theorem

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with interior CS, and ADR boundary. Then TFAE:

1. *$\partial\Omega$ is UR, and Ω satisfies "Weak Local John" (WLJ) condition.*
2. *Ω satisfies Interior Big Pieces of Chord-Arc Domains (IBPCAD).*
3. *$\omega \in$ weak-$A_\infty(\sigma)$.*

WLJ entails connected non-tangential path from CS point X_B to a "big piece" portion of $\Delta = B \cap \partial\Omega$; (could also be thought of as "Weak Local S-U").

Recent Results (continued)

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- (3) $\implies$ (1) has two parts: weak-A_∞ $\implies$ UR is S.H. - Martell result mentioned earlier; weak-A_∞ $\implies$ WLJ is new result of Azzam-Mourgoglou-Tolsa.

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Remark: direct proof (1) $\Longrightarrow$ (3) is slightly earlier result (a few months ago) of S.H. - Martell.

Remark: background hypotheses (upper and lower ADR, interior CS are in nature of best possible - $\exists$ C.E. in absence of any one of them.

Recent Results (continued)

Proof ingredients:

- (1) $\Longrightarrow$ (2): Corona approximation of UR set by CAD's (S.H. - Martell - Mayboroda 2016) plus 2-parameter bootstrapping scheme based on "extrapolation of Carleson measures" (J. Lewis).
- (3) $\Longrightarrow$ (1): (new part of [AMT]) use of Alt-Caffarelli-Friedman monotonicity formula to establish connectivity.

Thank you

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