Strong Comparison Principle for $p-$harmonic functions in Carnot-Caratheodory spaces

Xiaodan Zhou
Worcester Polytechnic Institute

Joint work with Luca Capogna

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Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open, connected set and $u : \Omega \to \mathbb{R}$.

$$-\Delta u = 0.$$

Strong Comparison Principle: Suppose $u, v \in C^2(\Omega) \cup C(\bar{\Omega})$ are harmonic in Ω. If $u \geq v$ in Ω, then either $u = v$ or $u > v$ in Ω.

Introduction

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For $1 \leq p < \infty$, the p-Laplace equation is defined by

$$L_p u = -div(|\nabla u|^{p-2} \nabla u) = 0.$$

Strong Comparison Principle remains open for $n \geq 3$.

See [Manfredi, '88] for the proof in $\mathbb{R}^2$.
See [Tolksdorf, '83] with extra condition that $|\nabla v| \geq \delta$ for some positive constant $\delta > 0$ in $\Omega \subset \mathbb{R}^n$.

Subelliptic p-Laplacian

Let $\Omega \subset \mathbb{R}^n$ be an open and connected set, and consider a family of smooth vector fields $X_1, \cdots, X_m$ in $\mathbb{R}^n$ satisfying Hörmander's finite rank condition,

$$\text{rank Lie}[X_1, \cdots, X_m](x) = n, \tag{1}$$

for all $x \in \Omega$. We set

$$Xu = (X_1 u, \cdots, X_m u).$$

We study the following class of equations

$$L_p u = \sum_{j=1}^{m} X_j^*(A_j(Xu)) = \sum_{j=1}^{m} X_j^*(|Xu|^{p-2} X_j u) = f(x, u), \tag{2}$$

where X_j^* denote the L^2 adjoint of the operator X_j with respect to the Lebesgue measure and we can write $X_j^* u = -X_j u - d_j(x) u$.

Structure conditions

The functions A_j ($A_j(\xi) = |\xi|^{p-2}\xi_j$) satisfy the following ellipticity and growth condition: For $p > 1$, for a.e. $\xi \in \mathbb{R}^m$ and for every $\eta \in \mathbb{R}^m$,

$$\begin{aligned} \sum_{i,j=1}^{m} \frac{\partial A_j}{\partial \xi_i}(\xi)\eta_i\eta_j &\geq \beta(\kappa + |\xi|)^{p-2}|\eta|^2 \\ \sum_{i,j=1}^{m} \left|\frac{\partial A_j}{\partial \xi_i}(\xi)\right| &\leq \gamma(\kappa + |\xi|)^{p-2} \end{aligned} \tag{3}$$

for some positive constants β, γ, and for $\kappa \geq 0$.

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$$\sum_{i,j=1}^{m} \frac{\partial A_j}{\partial \xi_i}(\xi)\eta_i\eta_j \geq \beta(\kappa + |\xi|)^{p-2}|\eta|^2$$
$$\sum_{i,j=1}^{m} |\frac{\partial A_j}{\partial \xi_i}(\xi)| \leq \gamma(\kappa + |\xi|)^{p-2} \tag{3}$$

for some positive constants β, γ, and for $\kappa \geq 0$. One can deduce that there exists positive constant λ, C such that for all $\xi \in \mathbb{R}^m$,

$$\langle A_j(\xi) - A_j(\xi'), \xi - \xi' \rangle \geq \begin{cases} \lambda(1 + |\xi| + |\xi'|)^{p-2}|\xi - \xi'|^2 & \text{if} \quad 1 \leq p \leq 2 \\ \lambda|\xi - \xi'|^p & \text{if} \quad 2 \leq p < \infty, \end{cases} \tag{4}$$

and

$$|A_j(\xi)| \leq C(\kappa + |\xi|)^{p-2}|\xi|. \tag{5}$$

A Weak Comparison Principle

The following lemma is an immediate consequence of the monotonicity inequality (4).

Lemma (Weak Comparison Principle)

Let $\Omega \subset \mathbb{R}^n$ be an open and connected set and $v_1, v_2 \in C^1(\Omega)$ satisfy in a weak sense

$$\begin{cases} L_p v_2 \le f(x, v_2) & \text{in } \Omega \\ L_p v_1 \ge f(x, v_1) & \text{in } \Omega, \end{cases}$$

with A_j satisfying the structure conditions (3) and $\partial_u f(x, u) \le 0$. If $v_2 \le v_1$ in $\partial\Omega$, then $v_2 \le v_1$ in Ω.

Strong Comparison Principle

In addition to the structure conditions (3), our strong comparison principle holds under the following hypothesis:

$$\begin{aligned} &(i) \quad \partial_u f \le 0 \text{ in } \Omega, \\ &(ii) \quad |f(x, u_2 + \epsilon) - f(x, u_2)| \le L\epsilon, \text{ for any } \epsilon \in [0, \epsilon_0], x \in \Omega \end{aligned} \tag{6}$$

Theorem (Strong Comparison Principle)

Let $\Omega \subset \mathbb{R}^n$ be a connected open set and consider two weak solutions $u_1 \in C^1(\bar{\Omega})$, and $u_2 \in C^2(\bar{\Omega})$ of (2) in Ω, with $|Xu_2| \ge \delta$ in Ω for some $\delta > 0$. We assume that the structure conditions (3), and the hypothesis (6) are satisfied. If

$$u_1 \ge u_2 \text{ in } \Omega,$$

then either $u_1 = u_2$ or

$$u_1 > u_2 \text{ in } \Omega.$$

Bony's Propagation Method

Definition

Let F be a relatively closed subset of Ω. We say that a vector $\mathbf{v} \in \mathbb{R}^n \setminus \{0\}$ is (exterior) normal to F at a point $y \in \Omega \cap \partial F$ if

$$\overline{B(y + \mathbf{v}, |\mathbf{v}|)} \subset (\Omega \setminus F) \cup \{y\}.$$

If this inclusion holds, we write $\mathbf{v} \perp F$ at y. Set

$$F^* = \{y \in \Omega \cap \partial F : \text{there exists } \mathbf{v} \text{ such that } \mathbf{v} \perp F \textit{ at } y\}.$$

Note when Ω is connected and $\emptyset \neq F \neq \Omega$, we have $F^* \neq \emptyset$.

Definition

Let X be vector field in Ω and $F \subset \Omega$ be a closed set. We say that X is tangent to F if, for all $x_0 \in F^*$ and all vectors v normal to F at x_0 one has that their Euclidean product vanishes, i.e. $\langle X(x_0), v\rangle = 0$.

Bony's Propagation Method

Theorem (Bony, 1969)

Let $\Omega \subset \mathbb{R}^n$ be an open set and $F \subset \Omega$ a closed subset. Let X be a Lipschitz vector field in Ω. Then X is tangent to F iff all its integral curves that intersect F are entirely contained in F.

Theorem (Bony, 1969)

Let $\Omega \subset \mathbb{R}^n$ be an open set and $F \subset \Omega$ a closed subset. Let $X_1, ..., X_m$ be smooth vector fields in Ω. If $X_1, ..., X_m$ are tangent to F then so is the Lie algebra they generate.

As a corollary, if $X_1, ..., X_m$ satisfy Hörmander finite rank condition (1) and are all tangent to F then every curve that touches F is entirely contained in F, so that either F is the empty set or $F = \Omega$.

A Hopf-type Comparison Principle

The key to the Strong Comparison Principle is the following result:

Lemma (A Hopf-type Comparison Principle)

Let $\Omega \subset \mathbb{R}^n$ be an open and connected set and $v_1 \in C^1(\Omega)$, $v_2 \in C^2(\Omega)$ with $|Xv_2| \geq \delta$ in Ω satisfy

$$\begin{cases} v_2 \leq v_1 & \text{in } \Omega \\ L_p v_2 \leq f(x, v_2) & \text{in } \Omega \\ L_p v_1 \geq f(x, v_1) & \text{in } \Omega. \end{cases} \tag{7}$$

Set $F = \{x \in \Omega : v_2(x) = v_1(x)\}$. If the structure conditions (3) and hypothesis (6) are satisfied and $\emptyset \neq F \neq \Omega$, then for every $y \in F^*$ and $\mathbf{v} \perp F$ at y, it follows that

$$\langle X_i(y), \mathbf{v} \rangle = 0$$

for all $i = 1, \cdots, m$.

Proof of the Hopf-type Comparison Principle

We argue by contradiction and suppose that there exists $y \in F^*$, a vector $\mathbf{v} \perp F$ at y, and $i \in \{1, \cdots m\}$ such that $\langle X_i(y), \mathbf{v} \rangle \neq 0$.

Let $z = y + \mathbf{v}$ and $r = |\mathbf{v}|$. We define

$$\sigma_i(x) := \langle X_i(x), x - z \rangle \qquad \sigma(x) = (\sigma_1(x), ..., \sigma_m(x)),$$

$$\tilde{b}(x) = e^{-\alpha|x-z|^2} \qquad b(x) = \alpha^{-2}(\tilde{b}(x) - e^{-\alpha r^2}).$$

Let V be a neighborhood of y and $U = V \cap B(z, r)$ and express its boundary as the union of two components

$$\partial U = \Gamma_1 \cup \Gamma_2,$$

where $\Gamma_1 = \overline{B(z, r)} \cap \partial V$ and $\Gamma_2 = \overline{V} \cap \partial B(z, r)$.

Proof of the Hopf-type Comparison Principle

By direct calculation and applying structure conditions (3), one obtains

$$L_p b(x) \leq -\tilde{b}(x)(\kappa + |Xb|)^{p-2}\left(4\beta|\sigma|^2 - 2\alpha^{-1}M_1\gamma\right).$$

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Further calculation combined with (6) and $|Xv_2| \ge \delta$ in Ω imply that

$$\begin{aligned}
L_p(b+v_2) &= -\sum_{i,j=1}^{m}\frac{\partial A_j}{\partial \xi_i}(X(b+v_2))X_jX_i(b+v_2) + d_jA_j(Xb+Xv_2)\\
&= -\sum_{i,j=1}^{m}\Big(\frac{\partial A_j}{\partial \xi_i}(X(b+v_2)) - \frac{\partial A_j}{\partial \xi_i}(Xv_2)\Big)X_jX_i(b+v_2)\\
&\quad -\sum_{i,j=1}^{m}\frac{\partial A_j}{\partial \xi_i}(Xv_2)X_jX_ib - \sum_{i,j=1}^{m}\frac{\partial A_j}{\partial \xi_i}(Xv_2)X_jX_iv_2 + d_jA_j(Xb+Xv_2)\\
&\le (-\epsilon_1 + M_2\epsilon_2\alpha^{-1} + C\alpha^{-1}|\sigma(x)|)\tilde{b}(x) + L|b| + f(x, b+v_2).
\end{aligned}$$

Proof of the Hopf-type Comparison Principle

By choosing α sufficiently large, one obtains

$$\begin{cases} v_2 + b \leq v_1 & \text{in } \partial U \\ L_p(v_2 + b) \leq f(x, b + v_2) & \text{in } U \\ L_p v_1 \geq f(x, v_1) & \text{in } U. \end{cases}$$

The Weak Comparison Principle in Lemma 1 implies that $v_2 + b \leq v_1$ in U. Finally, we observe that

$$\begin{aligned} 0 = \langle \mathbf{v}, \nabla(v_2 - v_1)(y) \rangle &= \lim_{t \to 0^+} \frac{v_2(y + t\mathbf{v}) - v_1(y + t\mathbf{v}) - (v_2(y) - v_1(y))}{t} \\ &\leq -\langle \mathbf{v}, \nabla b(y) \rangle \\ &= -2\alpha^{-1} r^2 e^{-\alpha r^2} < 0. \end{aligned}$$

Strong Maximum Principle

We also prove a non-homogenous strong maximum/minimum principle. We suppose that f satisfy the following conditions: for all $x \in \Omega$ and $u \in \mathbb{R}$,

$$\begin{aligned} &(i) \quad \partial_u f \leq 0, \\ &(ii) \quad |f(x,u)| \leq \bar{C}(\kappa + |u|)^{p-2}|u| \end{aligned} \tag{8}$$

for some positive constant $\bar{C}$ and κ as in the structure conditions (3).

Theorem (Strong Maximum Principle)

Let $\Omega \subset \mathbb{R}^n$ be a connected open set and consider a weak solution $u \in C^1(\bar{\Omega})$ of (2) in Ω. We assume that the structure conditions (3) and the hypothesis (8) hold. If

$$u \geq 0 \text{ in } \Omega,$$

then either $u = 0$ or

$$u > 0 \text{ in } \Omega.$$

Thank you!