Harmonic measure for sets of higher co-dimensions and BMO solvability

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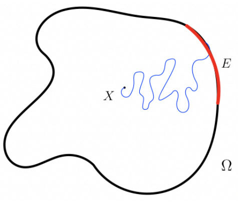

- For any $E \subset \partial\Omega$, its harmonic measure

$$\omega(E) = \mathbb{P}(\text{Brownian motion } B_t^X \text{ exits the domain } \Omega \text{ from } E).$$

- The solution to the Dirichlet problem

$$\begin{cases} -\operatorname{div}(A(X)\nabla u) &= 0, & \text{in } \Omega \\ u &= f, & \text{on } \partial\Omega \end{cases}$$

satisfies $u(X) = \int_{\partial\Omega} f \, d\omega_L^X$.

Question in co-dimension one: Given a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 3$), what is the relationship between the harmonic/elliptic measure ω and the surface measure $\sigma := \mathcal{H}^{n-1}|_{\partial\Omega}$?

In particular, what are the necessary and sufficient geometric assumptions to guarantee $\omega \ll \sigma \ll \omega$?

Sets of co-dimensions greater than 2 are invisible to harmonic measure!

Analogous harmonic measure for lower-dimensional sets:

- Lewis-Nyström 15'

  p-Laplace operator $-\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for $p > 2$

- David-Feneuil-Mayboroda 17'

  linear *degenerate* elliptic operator $-\operatorname{div}(A(X)\nabla)$

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Construction

- **Geometric assumption**: We consider the domain $\Omega = \mathbb{R}^n \setminus \Gamma$, where $\Gamma \subset \mathbb{R}^n$ is *d-Ahlfors regular* with $d < n-1$, that is, Γ is closed and

$$\mathcal{H}^d(\Gamma \cap B(q,r)) \sim r^d, \qquad \text{for any } q \in \Gamma, r > 0.$$

In this case we say $\sigma = \mathcal{H}^d|_\Gamma$ is the surface measure.

- **Analytic assumption**: We consider the operator $L = -\operatorname{div}(A(X)\nabla)$, where the matrix A satisfies

$$C_1 \delta(X)^{d-(n-1)} |\xi|^2 \leq A(X)\xi \cdot \xi \leq C_2 \delta(X)^{d-(n-1)} |\xi|^2,$$

for any $X \in \Omega$ and $\xi \in \mathbb{R}^n$. Here $\delta(X) = \operatorname{dist}(X, \Gamma)$.

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Construction (cont.)

Definition

There exists a bounded linear functional $U : C_c(\Gamma) \to C(\mathbb{R}^n)$ such that for any $f \in C_c(\Gamma)$, Uf solves the Dirichlet problem

$$\begin{cases} Lu = 0, & \Omega \\ u = f, & \Gamma \end{cases}$$

and satisfies the maximum principle.

Harmonic measure For any $X \in \Omega$ there exists a unique measure ω_Γ^X on Γ such that

$$Uf(X) = \int_\Gamma f \, d\omega_\Gamma^X.$$

Main Theorem

With the elliptic theory laid out in the work of David-Feneuil-Mayboroda 17', we are able to prove:

Theorem (Mayboroda-Z 18')

Under the above geometric and analytic assumptions,

$$\omega_\Gamma \in A_\infty(\sigma) \iff \text{the Dirichlet problem is solvable in BMO spaces.}$$

That is, for any $f \in C_c(\Gamma)$, *the solution* $u := Uf$ *satisfies* $|\nabla u|^2 \delta(X) dm(X)$ *is a Carleson measure, with norm bounded by* $\|f\|_{BMO}^2$.

Example of $\omega_\Gamma \in A_\infty(\sigma)$

Theorem (David-Feneuil-Mayboroda 17')

Suppose Γ is a Lipschitz graph on $\mathbb{R}^d$ with small Lipschitz constant, then the harmonic measure $\omega_\Gamma \in A_\infty(\sigma)$ for a specially chosen matrix $A(X)$ in the degenerate elliptic class.

Analogue in co-dimension one

Theorem (Dindos-Kenig-Pipher 09', Z 16'; co-dimension one)

Suppose $\Omega \subset \mathbb{R}^n$ *be a uniform domain with Ahlfors regular boundary. Let* $L = -\operatorname{div}(A(X)\nabla)$ *be a uniformly elliptic operator on* Ω, ω_L *be the corresponding elliptic measure and* $\sigma = \mathcal{H}^{n-1}|_{\partial\Omega}$.

$$\omega_L \in A_\infty(\sigma) \iff \text{the Dirichlet problem is solvable in BMO spaces.}$$

Ingredient of the proof in co-dimension one

Theorem (Dahlberg-Jerison-Kenig 84'; co-dimension one)

Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and $X_0 \in \Omega$ be fixed. Suppose the elliptic measure $\omega_L \in A_\infty(\sigma)$. Then for any $1 \leq p < \infty$, any solution u to $Lu = 0$ satisfying $u(X_0) = 0$, we have

$$C_1 \|Nu\|_{L^p(\sigma)} \leq \|Su\|_{L^p(\sigma)} \leq C_2 \|Nu\|_{L^p(\sigma)}.$$

Definition

We define the *non-tangential cone* with vertex $q \in \Gamma$ and aperture α as $\Gamma(q) = \{X \in \Omega : |X - q| < (1+\alpha)\delta(X)\}$. We define the non-tangential maximal function and square function as

$$Nu(q) = \sup_{X \in \Gamma(q)} |u(X)|, \qquad Su(q) = \left(\iint_{\Gamma(q)} |\nabla u|^2 \delta(X)^{1-d} dm(X) \right)^{\frac{1}{2}}$$

$$\|Su\|_{L^p(\sigma)} \leq C\|Nu\|_{L^p(\sigma)}$$

$$\uparrow$$

We can find $\delta = \delta(\epsilon)$ such that for all $\lambda > 0$

$$\sigma\left(\{q \in \partial\Omega : Su(q) > 2\lambda, Nu(q) \leq \delta\lambda\}\right) \leq \epsilon\sigma\left(\left\{q \in \partial\Omega : S'u(q) > \lambda\right\}\right)$$

$$\uparrow$$

$$\sigma\left(\{q \in \Delta : Su(q) > 2\lambda, Nu(q) \leq \delta\lambda\}\right) \leq \epsilon\sigma\left(\Delta\right), \text{ for some } \Delta$$

$$\uparrow$$

$$\omega^{X_\Delta}\left(\{q \in \Delta : Su(q) > 2\lambda, Nu(q) \leq \delta\lambda\}\right) \leq C\delta^2\omega^{X_\Delta}\left(\Delta\right)$$

Ingredient of the proof in higher co-dimensions

Theorem (Mayboroda-Z 18')

Let Γ be d-Ahlfors regular set in $\mathbb{R}^n$ with $d \leq n-1$. Suppose $\omega_\Gamma \in A_\infty(\sigma)$. Then for any $1 \leq p < \infty$, any solution $u \in W_r(\Omega)$ to $Lu = 0$, we have

$$\|Su\|_{L^p(\sigma)} \leq C\|Nu\|_{L^p(\sigma)}$$

as long as the right hand side is finite.

Thank you!