The Brunn-Minkowski inequality and a Minkowski problem for $\mathcal{A}$-harmonic Green's functions

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The question and the plan

This was joint work with Murat Akman, John Lewis, Olli Saari and just accepted in *Advances in Calculus of Variations*.

Background: For $1 < p < n$, in $\mathbb{R}^n$, using a generalization of p-capacity for operators $\mathcal{A}$ of p-Laplace type an $\mathcal{A}$-capacitary function (for E compact, convex with positive capacity) was used to solve the Brunn-Minkowski inequality and a Minkowski problem in *The Brunn-Minkowski inequality and a Minkowski problem for nonlinear capacity* with Akman, Gong, Hineman, Lewis, which followed Jerison and CNSXYZ.

Question: What to do when $n \leq p < \infty$ since p-capacity is no longer useful?

Plan: Try to generalize logarithmic capacity when $p = n$ to $n < p < \infty$ for p-Laplace type operators.

$\mathcal{A}$-harmonic equations, solutions

• For $p, \alpha \in (1, \infty)$, $\mathcal{A} : \mathcal{R}^n \setminus \{0\} \to \mathcal{R}^n$ belongs to the class $\mathcal{M}_p(\alpha)$ if it has continuous derivatives and satisfies ellipticity and homogeneity conditions

(i) $\frac{1}{\alpha}|\eta|^{p-2}|\boldsymbol{n}|^2 \leq \sum_{i,j=1}^n \frac{\partial \mathcal{A}_i}{\partial \eta_j}(\eta)\boldsymbol{n}_i\boldsymbol{n}_j$ and
$\sum_{i=1}^n |\nabla \mathcal{A}_i(\eta)| \leq \alpha|\eta|^{p-2}$
(ii) $\mathcal{A}(\eta) = |\eta|^{p-1}\mathcal{A}(\frac{\eta}{|\eta|})$, for all $\eta \neq 0$ set $\mathcal{A}(0) = 0$.

• $u \in W^{1,p}_{loc}(U)$ is $\mathcal{A}$-harmonic in open set $U \subset \mathcal{R}^n$ means: for all open G with $\overline{G} \subset U$

$$\int \mathcal{A}(\nabla u(y)) \cdot \nabla \theta(y)\, dy = 0 \text{ for all } \theta \in W^{1,p}_0(G)$$

shorthand version is $\nabla \cdot \nabla \mathcal{A}(\nabla u) = 0$

Examples of $\mathcal{A} \in \mathcal{M}_p(\alpha)$

- p-Laplace: Let $f(\eta) = \frac{1}{p}|\eta|^p$ set $\mathcal{A}(\eta) = \nabla f(\eta) = |\eta|^{p-2}\eta$ this gives the equation $\nabla \cdot |\nabla u|^{p-2}\nabla u = 0$

- Whenever f is p-homogeneous and $\mathcal{A}(\eta) = \nabla f(\eta)$, then the ellipticity condition on $\mathcal{A}$ says
$\eta \cdot D^2 f(\eta)\, \eta = p(p-1)f(\eta) \geq \frac{1}{\alpha}|\eta|^p$.

- For $f(\eta) = (1 + \frac{\epsilon\eta_1}{|\eta|})|\eta|^p$ with $\epsilon > 0$ small enough
$\mathcal{A}(\eta) = \nabla f(\eta)$ is <u>not</u> rotationally invariant.

- For u $\mathcal{A}$-harmonic on U and $T : V \to U$ a rotation then
$\tilde{u}(z) = u(Tz)$ is $\tilde{\mathcal{A}}$-harmonic in V where $\tilde{\mathcal{A}} \in \mathcal{M}_p(\alpha)$

In particular: if u is $\mathcal{A}$-harmonic then $1 - u$ is $\tilde{\mathcal{A}}$-harmonic where $\tilde{\mathcal{A}}(\eta) = -\mathcal{A}(-\eta)$, here $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are in the same class $\mathcal{M}_p(\alpha)$

The associated $\mathcal{A}$-harmonic measure

For $E \subset B(0,R)$ a nonempty, convex, compact set (containing at least two points when $p = n$) and $u > 0$ an $\mathcal{A}$-harmonic function in $B(0,4R) \setminus E$ with $u = 0$ on ∂E in an appropriate Sobolev sense there is a unique positive finite Borel measure ν with support in E associated to u so that

$$\int \mathcal{A}(\nabla u(y)) \cdot \nabla \phi(y)\, dy = -\int \phi\, d\nu \text{ for all } \phi \in C_0^\infty(B(0,2R))$$

In the harmonic case $p = 2$ and $\mathcal{A}(\eta) = \nabla \frac{1}{2}|\eta|^2$

$$d\nu(y) = 2|\nabla u(y)| dH^{n-1}(y)$$

In case $\mathcal{A}(\eta) = \nabla f(\eta)$ and there is enough regularity (Lemma 8.2)

$$d\nu(y) = p\frac{f(\nabla u(y))}{|\nabla u(y)|} dH^{n-1}(y)$$

Capacity when $p = n$?

- When $\mathcal{A}(\eta) = \nabla \frac{1}{p}|\eta|^p$ the p-capacity of a compact, convex set E with nonempty interior is

$$Cap_p(E) = \inf\{\int |\nabla v(x)|^p dx \mid v \in C_0^\infty(\mathbb{R}^n), v = 1 \text{ on } E\}$$

and the infimum is attained by a function u called the p-capacitary function. For $1 < p < n$ this is the function considered in the previous work on Brunn-Minkowski and Minkowski.

- For $p \geq n$ the p-capacity of any ball is 0, see HKM.

- Borell for $p = n = 2$, Colesanti and Salani for $p = n > 2$ consider the logarithmic capacity and study it in the Brunn-Minkowski inequality.

The plan for $p \geq n$ and $\mathcal{A} \in \mathcal{M}_p(\alpha)$

1. Get a $\mathcal{A}$-harmonic fundamental solution $F(x)$ with pole at 0.
2. For a compact convex set E (containing at least two points if $p = n$), get a $\mathcal{A}$-harmonic Green's function $G(x)$ with pole at infinity
3. Show that $G(x) = F(x) + k(x)$ and $k(\infty)$ exists. $k(\infty) \leq 0$ when $n < p < \infty$
4. for $p = n$ set $C(E) = e^{-k(\infty)/\gamma}$ then C is homogeneous of degree one
5. for $n < p < \infty$ set $C(E) = (-k(\infty))^{p-1}$ then $C^{\frac{1}{p-n}}$ is homogeneous of degree one
6. show Brunn-Minkowski for these 1-homogeneous set functions

More explicitly for $p = n$, $\mathcal{A} \in \mathcal{M}_n(\alpha)$

There is a unique, set $F(e_1) = 1$, fundamental solution with pole at 0, $F(x)$, satisfying
F is $\mathcal{A}$-harmonic in $\mathbb{R}^n \setminus \{0\}$

$$\int \mathcal{A}(\nabla F(x)) \cdot \nabla \theta(x) dx = -\theta(0) \text{ for all } \theta \in C_0^\infty(\mathbb{R}^n)$$

$\boxed{F(x) = \gamma \log |x| + b(x/|x|)}$ for $x \neq 0$, $\gamma > 0$, $b \in C^{1,\sigma}$ of a nhbd of the unit sphere, $\frac{1}{c} \leq \nabla F(z) \cdot z \leq |z||\nabla F(z)| \leq c$

Given a compact, convex set E containing at least two points there is a unique Green's function with pole at infinity $G(x)$ satisfying
G is $\mathcal{A}$-harmonic in E^c with continuous boundary value 0 on ∂E
$\boxed{G(x) = F(x) + k(x)}$, k bounded in a nhbd of infinity and $k(\infty)$ exists, $|k(x) - k(\infty)| \leq \hat{r}_0 |x|^{-\beta}$, $|x| \geq \hat{r}_0$

$C(E)$ for $p = n$ is 1-homogeneous

Define $C(E) = e^{-k(\infty)/\gamma}$ note that C is homogeneous of degree one. Write G_E, k_E for the Green's function on E^c with pole at ∞, consider

$$\begin{aligned} G_E(x/t) &= F(x/t) + k_E(x/t) \\ &= \gamma \log |x/t| + b(x/|x|) + k_E(x/t) \\ &= \gamma \log |x| + b(x/|x|) + k_E(x/t) - \gamma \log t \end{aligned}$$

This is the Green's function for tE, with $k(\infty) = k_E(\infty) - \gamma \log t$ so that

$$C(tE) = e^{(-k_E(\infty) - \gamma \log t)/\gamma} = tC(E)$$

More explicitly for $p > n$, $\mathcal{A} \in \mathcal{M}_p(\alpha)$

There is a unique fundamental solution $F(x)$ with pole at ∞, satisfying F is $\mathcal{A}$-harmonic in $\mathbb{R}^n \setminus \{0\}$, $F(0) = 0$, $F(x) > 0$ for $x \neq 0$,

$$\int \mathcal{A}(\nabla F(x)) \cdot \nabla \theta(x) dx = -\theta(0) \text{ for all } \theta \in C_0^\infty(\mathbb{R}^n)$$

$\boxed{F(x) = |x|^{\frac{p-n}{p-1}} \psi(x/|x|)}$ where ψ is $C^{1,\sigma}$ on the unit sphere.
$\frac{1}{c} F(z) \leq \nabla F(z) \cdot z \leq |z||\nabla F(z)| \leq cF(z)$
Given a nonempty compact, convex set E there is a unique $\mathcal{A}$-harmonic Green's function on E^c with pole at ∞ and continuous boundary value 0 on ∂E satisfying
$\boxed{G(x) = F(x) + k(x)}$ where $k(x)$ is bounded in a nbhd of ∞ and $k(\infty)$ exists, $|k(x) - k(\infty)| \leq \hat{r}_0 |x|^{-\beta}$, $|x| \geq \hat{r}_0$

$C(E)$ for $p > n$ is $p - n$ homogeneous

Define $C(E) = (-k(\infty))^{p-1}$ let's show that C is homogeneous of degree $p - n$. Write G_E for the Green's function of E with pole at ∞, consider

$$\begin{aligned} t^{\frac{p-n}{p-1}} G_E(x/t) &= t^{\frac{p-n}{p-1}} (F(x/t) + k_E(x/t)) \\ &= t^{\frac{p-n}{p-1}} (|x/t|^{\frac{p-n}{p-1}} \psi(x/|x|) + k_E(x/t)) \\ &= |x|^{\frac{p-n}{p-1}} + t^{\frac{p-n}{p-1}} k_E(x/t) \end{aligned}$$

So this is Green's function for tE with pole at ∞, $k(\infty) = t^{\frac{p-n}{p-1}} k_E(\infty)$ and

$$C(tE) = (-t^{\frac{p-n}{p-1}} k_E(\infty))^{p-1} = t^{p-n} C(E)$$

the Brunn-Minkowski inequality: for all E_1, E_2 compact, convex sets (with at least two points when $p = n$) for all $\lambda \in (0, 1)$

When $p = n$

$$C((1-\lambda)E_1 + \lambda E_2) \geq (1-\lambda)C(E_1) + \lambda C(E_2)$$

When $p > n$

$$C((1-\lambda)E_1 + \lambda E_2)^{\frac{1}{p-n}} \geq (1-\lambda)C(E_1)^{\frac{1}{p-n}} + \lambda C(E_2)^{\frac{1}{p-n}}$$

By clever choices of sets and parameters these are equivalent to

$$C((1-\lambda)E_1 + \lambda E_2) \geq \min\{C(E_1), C(E_2)\}$$

Proof, convert this situation, Green's functions, to the one in the previous paper, capacitary functions.

The Brunn-Minkowski Theorem

Theorem A. *Let E_1 and E_2 be compact convex sets in $\mathbb{R}^n, n \geq 2$. Assume that both sets contain at least two points when $p = n$ and that both sets are nonempty when $p > n$. If $\lambda \in [0, 1]$ and if $p = n$ then*

$$\mathcal{C}_{\mathcal{A}}(\lambda E_1 + (1-\lambda)E_2) \geq \lambda \mathcal{C}_{\mathcal{A}}(E_1) + (1-\lambda)\mathcal{C}_{\mathcal{A}}(E_2). \tag{2.4}$$

While if $n < p < \infty$ then

$$[\mathcal{C}_{\mathcal{A}}(\lambda E_1 + (1-\lambda)E_2)]^{\frac{1}{p-n}} \geq \lambda \mathcal{C}_{\mathcal{A}}(E_1)^{\frac{1}{p-n}} + (1-\lambda)\mathcal{C}_{\mathcal{A}}(E_2)^{\frac{1}{p-n}}. \tag{2.5}$$

If equality holds in (2.4) or in (2.5) and $\mathcal{A}$ satisfies

(2.6)

(*i*) *There exists $1 \leq \Lambda < \infty$ such that* $\left| \frac{\partial \mathcal{A}_i}{\partial \eta_j}(\eta) - \frac{\partial \mathcal{A}_i}{\partial \eta'_j}(\eta') \right| \leq \Lambda \, |\eta - \eta'| |\eta|^{p-3}$

whenever $0 < \frac{1}{2}|\eta| \leq |\eta'| \leq 2|\eta|$ *and* $1 \leq i \leq n$,

(*ii*) $\mathcal{A}_i(\eta) = \frac{\partial f}{\partial \eta_i}$ *for* $1 \leq i \leq n$ *where* $f(t\eta) = t^p f(\eta)$ *when* $t > 0$ *and* $\eta \in \mathbb{R}^n \setminus \{0\}$,

then E_2 is a translation and dilation of E_1 provided that both sets contain at least two points.

Equality in Brunn-Minkowski

This relies on some ideas of Colesanti and Salani.
$f(\eta) = (k(\eta))^p$, k is 1-homogeneous, k^2 is strictly convex
Set $B_k = \{\eta \mid k(\eta) \leq 1\}$ and let $h(X) = \sup_{\eta \in B_k} X \cdot \eta$ be the support function, it's 1-homogeneous.
Then $k\nabla k$ and $h\nabla h$ are inverses of each other on $\mathbb{R}^n \setminus \{0\}$ and

$$\hat{F}(X) = \begin{cases} h(X)^{\frac{p-n}{p-1}} & n < p < \infty \\ \log h(X) & p = n \end{cases}$$

is a constant multiple of the fundamental solutions above! See remark 6.3.

Hadamard formula, Proposition 10.1 Remark 10.2

For convex compact sets E_1, E_2 with $0 \in E_1$, (not necessarily $0 \in E_1^\circ$) and $0 \in E_2^\circ$, and $t \geq 0$ we have $n < p < \infty$

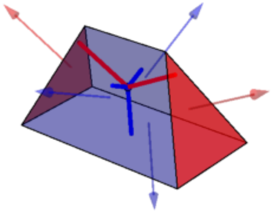
$$\frac{d}{dt} C(E_1 + tE_2) = p(p-1) C(E_1 + tE_2)^{\frac{p-2}{p-1}} \int_{\partial(E_1+tE_2)} h_2(g(x)) f(\nabla u(x)) dH^{n-1}$$

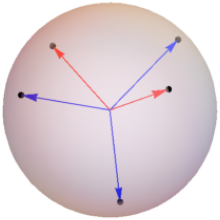
h_2 is the support function of E_2, g is the Gauss map of $E_1 + tE_2$ and u is the $\mathcal{A}$-harmonic Green's function of $E_1 + tE_2$. While for $p = n$ this is

$$\frac{d}{dt} C(E_1 + tE_2) = \frac{n}{\gamma} C(E_1 + tE_2) \int_{\partial(E_1+tE_2)} h_2(g(x)) f(\nabla u(x)) dH^{n-1}$$

note that Brunn-Minkowski says $C(E_1 + tE_2)^{\frac{1}{p-n}}$ or $C(E_1 + tE_2)$ are concave.

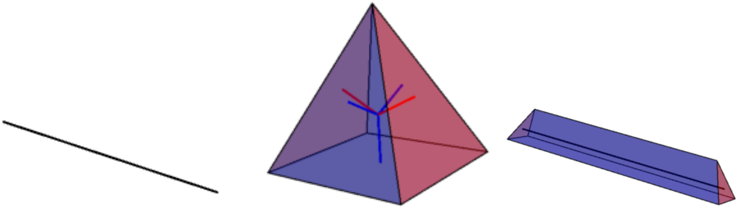
Polyhedron, Gauss map, support function.





Gauss map: 2 red faces (right, left) and 3 blue faces (front, bottom $= F_1$, back) for $x \in F_1$, $g(x) = -e_3$, $g^{-1}(-e_3) = F_1$.
Support function: for $x \in$ bottom face, $h(g(x))$ is the distance of the face to the origin, the length of the vertical thick blue segment.
Next Slide: Move the 3 blue faces to the origin, the solid blue segments shrink to zero, call this E_1. Make all the solid segments the same length, call this E_2.

Polyhedron example E_1, E_2 and $E_1 + tE_2$



- E_2 has five unit normals $\boldsymbol{n}_1, \ldots, \boldsymbol{n}_5$ all with $h_2(\boldsymbol{n}_k) = a$

On the faces F_i, $i = 1, \ldots, 5$ of $E_1 + tE_2$ the integrals above for $\frac{d}{dt}C(E_1 + tE_2)$ are

$$\begin{cases} p(p-1)C(E_1+tE_2)^{\frac{p-2}{p-1}} \sum_{i=1}^{5} a \int_{F_i} f(\nabla u(x))dH^{n-1} & n < p < \infty \\ \frac{n}{\gamma}C(E_1+tE_2) \sum_{i=1}^{5} a \int_{F_i} f(\nabla u(x))dH^{n-1} & p = n \end{cases}$$

Hadamard capacity formula

In case $E_1 = E_2 = E_0$ and $t = 0$ using the homogeneity of $C((1+t)E_0)$ and taking the derivative at $t = 0$ we have for $n < p < \infty$

$$\frac{(p-n)}{(p-1)} C(E_0)^{\frac{1}{p-1}} = p \int_{\partial E_0} h(g(x)) f(\nabla u(x)) dH^{n-1}$$

Where h, g and u are the support, Gauss, and $\mathcal{A}$-harmonic Green's functions for E_0.
While for $p = n$

$$\gamma = n \int_{\partial E_0} h(g(x)) f(\nabla u(x)) dH^{n-1}$$

For a polyhedron, Hadamard capacity formula

For E_0 a polyhedron with $0 \in E_0^\circ$, with faces F_k with unit outer normals $\boldsymbol{n}_k$ and distance to the origin q_k this gives (say $k = 1, \ldots, m$) for $n < p < \infty$

$$\frac{(p-n)}{(p-1)} C(E_0)^{\frac{1}{p-1}} = \sum_{i=1}^{m} \int_{F_i} h(\boldsymbol{n}_i) f(\nabla u) dH^{n-1} = \sum_{i=1}^{m} q_i c_i$$

where $c_i = \int_{F_i} f(\nabla u) dH^{n-1}$ think of this as the mass of each face.

While for $p = n$

$$\gamma = n \sum_{i=1}^{m} q_i c_i$$

C is Translation invariant

Translating E_0 by x, then the equations above are invariant, except that the support function of $E_0 + x$ is $h(\boldsymbol{n}) + x \cdot \boldsymbol{n}$ so that

$$\sum_{i=1}^{m} q_i c_i = \sum_{i=1}^{m} (q_i + x \cdot \boldsymbol{n}_i) c_i$$

which gives, for all x,

$$\sum_{i=1}^{m} (x \cdot \boldsymbol{n}_i) c_i = 0$$

and therefore

$$\sum_{i=1}^{m} \boldsymbol{n}_i c_i = 0$$

The Minkowski Theorem

The function $\mathbf{g}(\cdot, E) : \partial E \mapsto \mathbb{S}^{n-1}$ (whenever defined) is called the Gauss map for ∂E. Let μ be a finite positive Borel measure on $\mathbb{S}^{n-1}$ satisfying

$$
(7.1) \quad \begin{aligned} &(i) \int_{\mathbb{S}^{n-1}} |\langle \theta, \zeta \rangle| \, d\mu(\zeta) > 0 \quad \text{for all } \theta \in \mathbb{S}^{n-1}, \\ &(ii) \int_{\mathbb{S}^{n-1}} \zeta \, d\mu(\zeta) = 0. \end{aligned}
$$

We prove

Theorem B. *Let μ be as in (7.1) and p be fixed, $n \leq p < \infty$. Let $\mathcal{A} = \nabla f$ be as in (2.6) and Definition 2.1. Then there exists a compact convex set E with nonempty interior and $\mathcal{A}$-harmonic Green's function u for $\mathbb{R}^n \setminus E$ with a pole at infinity satisfying*

(7.2)

(a) $\lim_{y \to x} \nabla u(y) = \nabla u(x)$ *exists for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$ as $y \in \mathbb{R}^n \setminus E$ approaches x non-tangentially.*

(b) $\displaystyle\int_{\partial E} f(\nabla u(x)) \, d\mathcal{H}^{n-1} < \infty.$

(c) $\displaystyle\int_{\mathbf{g}^{-1}(K,E)} f(\nabla u(x)) \, d\mathcal{H}^{n-1} = \mu(K)$ *whenever $K \subset \mathbb{S}^{n-1}$ is a Borel set.*

(d) *E is the unique set up to translation for which (c) holds.*

The Minkowski theorem in the discrete case, existence.

Let μ be a finite positive Borel measure on the unit sphere $\mathbb{S}^{n-1}$ with masses $c_i > 0$ at distinct points $\boldsymbol{n}_i$ for $i = 1, \ldots, m$.
If (i) $\sum_{i=1}^m c_i |\theta \cdot \boldsymbol{n}_i| > 0$ for all $\theta \in \mathbb{S}^{n-1}$ and (ii) $\sum_{i=1}^m c_i \boldsymbol{n}_i = 0$ then there is a compact, convex, set E_0 with nonempty interior so that

$$\mu(K) = \int_{g^{-1}(K)} f(\nabla u) dH^{n-1}$$

where g and u are the Gauss and $\mathcal{A}$-harmonic Green's functions for E_0.
E_0 is unique up to translation.
Generally folks assume no antipodal point masses at this point, it rules out getting $n-1$ sets in the upcoming minimization.
Later on these sets are considered in the continuous measure case.

The minimization procedure

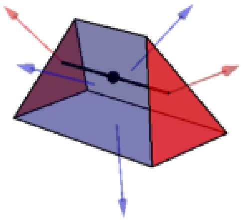
For $q_i \geq 0$ let

$$
\begin{aligned}
E(q) &= \bigcap_{i=1}^{m} \{x \mid x \cdot \boldsymbol{n}_i \leq q_i\} \text{ intersection of half spaces} \\
\Phi &= \{E(q) \mid C(E(q)) \geq 1\} \text{ with capacity } \geq 1 \\
\lambda(q) &= \sum_{i=1}^{m} q_i c_i \text{ Hadamard capacity formula} \\
\lambda &= \inf_{E(q) \in \Phi} \lambda(q) \text{ minimize it}
\end{aligned}
$$

Because of condition (i) the $E(q) \in \Phi$ are bounded, compact, convex sets.
There is a sequence $q^k \to \hat{q}$ so that $E(q^k) \to E(\hat{q}) = E_1$ a convex, compact set with $\lambda = \lambda(\hat{q})$
Is E_1° nonempty? Do we have $\hat{q}_i > 0$ for $i = 1, \ldots, m$?

Recall the examples

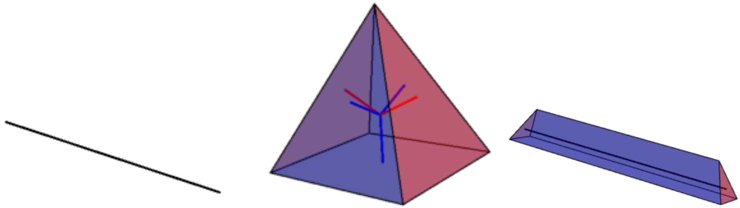
- Imagine the 3 blue faces moving to the origin and giving the minimizer E_1 as the black 1-d segment. The $\hat{q}_i$ for the blue faces are all 0.



- Or imagine that the two red faces are parallel and that they move to the origin, giving a 2-d set for the minimizer E_1. The $\hat{q}_i$ for the red faces are now 0. But this is ruled out by no antipodal masses!
- In either case, maybe $C(E_1) = 1$ is possible!

The minimizer E_1 has nonempty interior $p > n$

For $1 \leq \dim(E_1) < n - 1$, a situation illustrated here



We set $E_2 = \bigcap_{i=1}^{m} \{x \mid x \cdot \boldsymbol{n}_i \leq a\}$ and consider $E_1 + tE_2$
It turns out that we can study a $k(t)$ with $k(0) = \lambda$ and $\lambda(q(t)) \leq k(t) < \lambda$ for $t > 0$ close to zero
This contradicts λ being the minimum, so this situation does not occur!

The minimizer E_1 has nonempty interior, $p > n$

Here's $k(t) = C(E_1 + tE_2)^{-1/(p-n)} \sum_{i=1}^{m} c_i(\hat{q}_i + at)$
taking the derivative we get a negative term involving the derivative of the capacity which blows up as $t \to 0^+$ (hence k is decreasing and $k(t) < k(0) = \lambda$ for $t \to 0^+$).

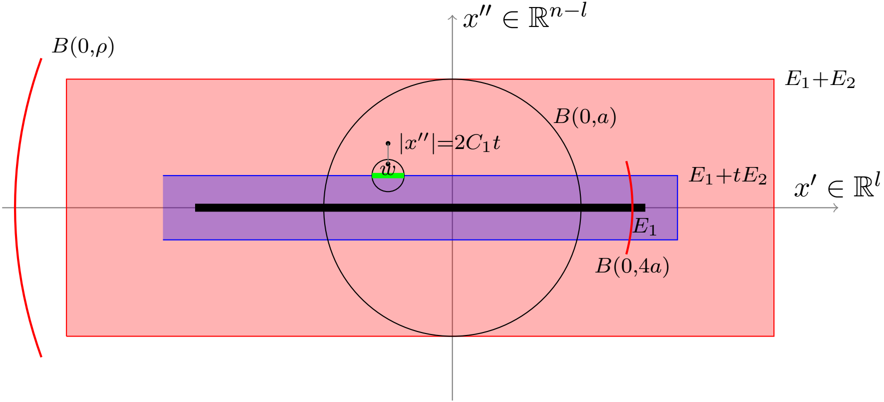
$$\lim_{t \to 0^+} \int_{\partial(E_1 + tE_2)} h_2(g(x,t)) f(\nabla u(x,t)) dH^{n-1} = \infty$$

where g and u are the Gauss and $\mathcal{A}$-harmonic Green's functions of $E_1 + tE_2$ and h_2 is the support function of E_2 so always $\geq a$ so we only need to show

$$\lim_{t \to 0} \int_{\partial(E_1 + tE_2)} f(\nabla u(x,t)) dH^{n-1} = \infty$$

the argument for $1 \leq dim(E_1) \leq n-2$

A schematic for Equations (11.26) to (11.29)



notes: $dS = dH^{n-1}$, $B(0,4a) \cap \mathbb{R}^l \subset E_1$, $|\Delta| \approx t^{n-1}$, $u = u(x,t)$

$$t^{n-p}u(w)^{p-1} \approx \nu(\Delta) \approx \int_{\Delta} \frac{f(\nabla u)}{|\nabla u|} dS \leq \left(\int_{\Delta} f(\nabla u) dS \right)^{\frac{p-1}{p}} t^{\frac{n-1}{p}}$$

finishing... $1 \le dim(E_1) \le n-2$

Let $\psi = \frac{p-(n-l)}{p-1}$. Lemma 11.2 says $u(x,t) \gtrapprox |x''|^{\psi}$ for $t \lessapprox |x''|$, then Harnack $u(w,t) \approx u(x', x'', t) \gtrapprox t^{\psi}$ and some arithmetic gives

$$t^{p(\psi-1)+n-1} \lessapprox \int\limits_{\Delta} f(\nabla u(x,t))dS$$

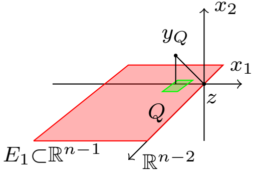
There are at least t^{-l} disjoint such Δ giving

$$t^{\frac{l+1-n}{p-1}} \lessapprox \int_{\partial(E_1+tE_2)} f(\nabla u(x,t))dS$$

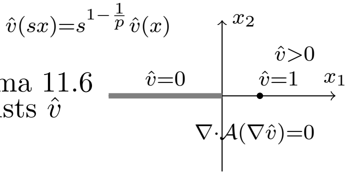
Now $1 \le l \le n-2$ so $l+1-n \le -1$ showing that this integral blows up as $t \to 0$.

For $dim(E_1) = n - 1$

A schematic for (11.68) to (11.71), let the sidelength of $Q = Q(\tau)$ be τ, then $\operatorname{dist}(y_Q, z) \approx \operatorname{dist}(y_Q, Q) \approx \operatorname{dist}(Q, z) \approx \tau$.



By Lemma 11.6 there exists $\hat{v}$



$u(x) \geq v(x)$ in $B(0, 2\rho)$ where $v(x) = \hat{v}(x_1 - z_1, x_2 - z_2)$. Then, as above,

$$\int_Q f(\nabla u_+) dS \gtrapprox u^p(y_Q)\tau^{n-1-p} \gtrapprox v^p(y_Q)\tau^{n-1-p} \gtrapprox \tau^{p-1+n-1-p}$$

or

$$\int_Q f(\nabla u_+) dS \gtrapprox \tau^{n-2}$$

finishing... $dim(E_1) = n - 1$

For each large integer l the number of cubes $Q(\tau_l)$ with sidelength $2^{-l-1}a \leq \tau_l \leq 2^{-l}a$ is at least $\gtrapprox 2^{l(n-2)}$ so that

$$\int_{E_1} f(\nabla u_+) dS \gtrapprox \sum_{l>N_0} \sum_{Q(\tau_l)} 2^{l(n-2)} \tau_l^{n-2}$$

$$\gtrapprox \sum_{l>N_0} 2^{l(n-2)-(l+1)(n-2)}$$

$$\gtrapprox \sum_{l>N_0} 2^{2-n} = \infty$$

John's Talk

What about solutions in the complement of a ray in higher dimensions, what can you say about the homogeneity?

Answers could lead to regularity in the Minkowski problem.

That's John's talk, Next!