The obstacle problem for the fractional heat equation: properties of the free boundary.

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Thank you for the invitation!

In this talk we will discuss the structure of the free boundary in the obstacle problem for the fractional heat equation.

Our goal is to provide a systematic classification of free boundary points based on the blowup limits of non-homogeneous Almgren-Poon type rescalings.

We also establish new monotonicity formulas of Weiss- and Monneau-type, which we employ to establish a structure theorem for the singular set of the free boundary.

This is joint work with Agnid Banerjee (TIFR), Nicola Garofalo (University of Padova), and Arshak Petrosyan (Purdue University).

Outline

- Motivation
- Statement of the problem
- Related Results
- Regularity of Solutions
- Monotonicity formulas and the study of the free boundary

The Fractional Heat Operator

Our goal is to study the structure of the free boundary for the nonlocal obstacle problem

$$\min\{u-\psi,\,(\partial_t-\Delta)^s u\}=0.$$

The function ψ is the *obstacle*, and

$$(\partial_t-\Delta)^s u(x,t)=$$
$$=\frac{s}{\Gamma(1-s)}\int_{-\infty}^{t}\int_{\mathbb{R}^n}(t-\tau)^{-s-1}G(x-z,t-\tau)[u(x,t)-u(z,\tau)]dzd\tau$$

denotes the *fractional heat operator*.

Here $0<s<1$, $u\in C^1(\mathbb{R}^n\times\mathbb{R})\cap L^\infty(\mathbb{R}^n\times\mathbb{R})$, $G(z,\tau)=(4\pi\tau)^{-\frac{n}{2}}e^{-\frac{|z|^2}{4\tau}}$ is the standard heat kernel and $\Gamma(z)$ is Euler Gamma function.

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Motivation

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An important motivation for the study of this nonlocal operator comes from the fact that it models a stochastic jump process with arbitrary distributions of both jump lengths and waiting times, such as the *continuous time random walk* (CTRW) introduced by Montroll and Weiss (1965).

The stationary case

The elliptic counterpart

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In 2007 Caffarelli and Silvestre introduced a remarkable extension procedure which allows to convert problems involving the fractional Laplacian $(-\Delta)^s$ acting on functions of $x \in \mathbb{R}^n$, into ones involving a **local degenerate elliptic** operator acting on functions of the variable $X = (x, y) \in \mathbb{R}^{n+1}_+ = \mathbb{R}^n_x \times \mathbb{R}^+_y$.

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This method opened the way to an exhaustive study of the regularity properties of both the solution and the free boundary for the time-independent obstacle problem for all $0<s<1$.

Statement of the Problem

Nystrom and Sande (2016) and - indipendently - Stinga and Torrea (2017) showed that, at a local level, the nonlocal obstacle problem

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This is a *thin obstacle problem* since now the obstacle ψ lives on the thin manifold Q_1 in space-time $\mathbb{R}^n\times(-1,0)$.

Notations:

- $\mathbb{B}_r$ = thick ball in the thick variable $X \in \mathbb{R}^{n+1}$
- B_r = thin ball in the thin variable $x \in \mathbb{R}^n$.
- $\mathbb{Q}_r = \mathbb{B}_r \times (-r^2, 0]$ = thick parabolic cylinder in the thick space $(X, t) \in \mathbb{R}^{n+1} \times \mathbb{R}$
- $Q_r = B_r \times (-r^2, 0]$ = thin parabolic cylinder in the thin space $(x, t) \in \mathbb{R}^n \times \mathbb{R}$
- $\mathbb{B}_r^+ = \{X = (x, y) \in \mathbb{B}_r \mid y > 0\}$ = thick half-ball
- $\mathbb{Q}_r^+ = \mathbb{B}_r^+ \times (-r^2, 0]$ = thick parabolic half-cylinder
- $\mathbb{S}_r = \{(X, t) | X \in \mathbb{R}^{n+1}, \quad -r^2 < t < 0\}$ = strip in thick space
- $S_r = \{(x, t) \mid x \in \mathbb{R}^n, \quad -r^2 < t < 0\}$ = strip in thin space
- $\mathbb{S}_r^+ = \{(X, t) \mid X \in \mathbb{R}_+^{n+1}, \quad -r^2 < t < 0\}$ = half-strip in thick space

The pioneering work of Chiarenza and Serapioni

The equation $\mathcal{L}_a V = 0$ is a special case of the class of degenerate parabolic equations in divergence form

$$\partial_t(\omega(X)V) = \mathrm{div}(A(X)\nabla V),$$

where $\omega(X)$ is a Muckenhoupt A_2-weight which controls the degeneracy of the matrix-valued function $A(X)$.

These equations were first studied by Chiarenza and Serapioni (1985).

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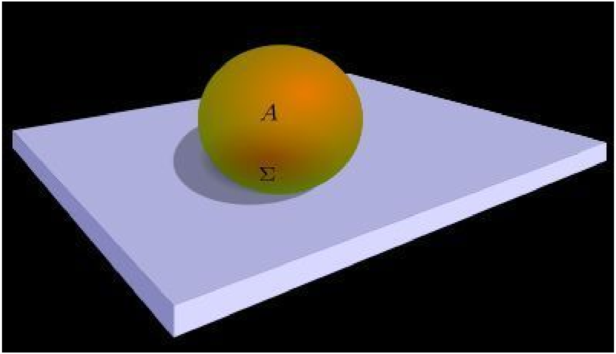
If we take $\omega(X) = |y|^a$, with $a = 1 - 2s$, then we have $\omega \in A_2(\mathbb{R}^{n+1})$ since $|a| < 1$.
Using the Chiarenza-Serapioni result and the Signorini conditions we know that local solutions to the thin obstacle problem satisfy a parabolic Harnack inequality and are therefore Hölder continuous up to the thin set $(\mathbb{R}^n \times \{0\}) \times (-1, 0)$.

The Signorini Problem

The case $a = 0$ corresponds to the Signorini problem:

What is the equilibrium configuration of an elastic body resting on a rigid frictionless plane?



Other applications include:

- Optimal control of temperature across a surface
- Modeling of semipermeable membranes (osmosis)
- Probability and financial math (optimal stopping problems for stochastic processes with jumps)
- Geophysical fluid dynamics (quasi-geostrophic equations)

Related Results

In joint work with Garofalo, Petrosyan and To (2017), we proved

- Existence and homogeneity properties of blow-up limits (by means of a monotonicity formula of Almgren-Poon type)
- Optimal regularity of solutions
- Classification of free boundary points
- Regularity of the regular set
- Structure of the singular set (by means of monotonicity formulas of Weiss- and Monnau type)

The parabolic nonlocal obstacle problem

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has been treated by Caffarelli and Figalli (2013) and Barrios, Figalli, and Ros-Oton (2018). However, even if the stationary versions are the same, this problem is fundamentally different from the one we are considering.

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In recent work Athanasopoulos, Caffarelli and Milakis (2018) establish the optimal regularity of solutions, as well as $C^{1,\alpha}$-regularity of the free boundary at certain non-singular points for solutions to

$$\min\big\{u - \psi,\, (\partial_t - \Delta)^s u\big\} = 0,$$

using the correspondence with the local degenerate problem.

Reduction to zero obstacle and globalization

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This is accomplished by subtracting the obstacle, and multiplying by a cut-off $\zeta(X) = \zeta^{\star}(|X|) \in C_0^{\infty}(\mathbb{B}_1)$, $0 \le \zeta \le 1$, and then considering the new function

$$U(X,t) = \zeta(X)(V(X,t) - \psi(x,t)).$$

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Important observation: Since ζ is smooth and $\zeta(x,-y) = \zeta(x,y)$ one has

$$\lim_{y\to 0^+} y^a \frac{\partial V}{\partial y}(x,y,t) = \lim_{y\to 0^+} y^a \frac{\partial U}{\partial y}(x,y,t).$$

Therefore, the function U solves the following problem in the space-time strip $\mathbb{S}_1^+$ in thick space

$$\begin{cases} \mathcal{L}_a U = y^a F & \text{in } \mathbb{S}_1^+, \\ & \\ & \end{cases}$$

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$$\begin{cases} \mathcal{L}_a U = y^a F & \text{in } \mathbb{S}_1^+, \\ U(x,0,t) \geq 0, & \text{for } (x,t) \in S_1^+, \\ & \\ & \end{cases}$$

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If $\psi \in C^2_{x,t}$, then not only $F \in L^\infty(\mathbb{S}_1^+)$ but also $F_t \in L^\infty(\mathbb{S}_1^+)$!
This allows us to prove the crucial fact

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With this information we can bring $y^a U_t$ to the right-hand side and then, setting $F - U_t \longrightarrow F$, consider the *elliptic* problem for the function $u(X) = U(X, \bar{t})$ **at each fixed time-level** $\bar{t}$:

$$\begin{cases} \operatorname{div}_X(y^a \nabla_X u) = y^a F & \text{in } \mathbb{B}_1^+, \\ u(x,0) \geq 0, & \text{for } x \in B_1^+, \\ -\lim\limits_{y \to 0^+} y^a \frac{\partial u}{\partial y}(x,y) \geq 0, & \text{for } x \in B_1^+, \\ \lim\limits_{y \to 0^+} y^a \frac{\partial U}{\partial y}(x,y) = 0, & \text{on the set } \{x \in B_1^+ \mid u(x,0) > 0\} \end{cases}$$

Regularity of Solutions

Using the *elliptic* regularity results of Caffarelli, De Silva and Savin (2017), and the fact that the estimates are uniform in $\bar{t} \in (-1, 0)$, we prove that

$$\nabla_x U \in \mathbb{H}^{\frac{1-a}{2}, \frac{1-a}{4}}(\mathbb{S}_1^+ \cup (S_1 \times \{0\}))$$

($\mathbb{H}^{\alpha,\alpha/2}$ = intrinsic parabolic Hölder classes)
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In addition, thanks to some delicate $W^{2,2}(\mathbb{Q}_1^+, y^a dXdt)$ estimates, we show that

$$|\nabla U_{x_i}|^2 \in L^2(\mathbb{S}_1^+, y^a dXdt).$$

The free boundary: Preliminaries

Denote by

$$\overline{\mathcal{G}}_a(X,t) = \mathcal{G}_a(X,0,|t|), \qquad\qquad t<0,$$

the Neumann fundamental solution of the backward operator

$$\mathcal{L}_a^\star = y^a \frac{\partial}{\partial t} + \operatorname{div}_X(y^a \nabla_X)$$

with pole at $0=(0,0,0)$. One has the remarkable formula

$$\overline{\mathcal{G}}_a(X,t) = \frac{(4\pi)^{-\frac{n}{2}}}{2^a \Gamma(\frac{a+1}{2})} |t|^{-\frac{n+a+1}{2}} e^{-\frac{|X|^2}{4|t|}}.$$

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We now introduce the quantities

$$H(U,r) = \frac{1}{r^2}\int_{\mathbb{S}_r^+} U^2\, \overline{\mathcal{G}}_a y^a dXdt, \quad D(U,r) = \frac{1}{r^2}\int_{\mathbb{S}_r^+} |t||\nabla U|^2\, \overline{\mathcal{G}}_a y^a dXdt.$$

One-parameter Almgren-Poon type monotonicity formula

One of our main tools is the following result:

Theorem 1 (Truncated monotonicity formula of Almgren-Poon type)

Suppose that $|F(X,t)| \leq C_\ell |(X,t)|^{\ell-2}$ *for every* $(X,t) \in \mathbb{S}_1^+$*, for* $\ell \geq 2$ *and some constant* $C_\ell > 0$*. Then, for every* $\sigma \in (0,1)$ *there exist a constant* $C > 0$*, depending on* n, a, C_ℓ *and* σ*, such that the function*

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$$r \mapsto \Phi_{\ell,\sigma}(U,r) \overset{def}{=} re^{Cr^{1-\sigma}} \frac{d}{dr} \log \max \left\{ H(U,r), \ r^{2\ell-2+2\sigma} \right\} + 4(e^{Cr^{1-\sigma}} - 1),$$

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is monotone nondecreasing on $(0,1)$. *In particular, it exists*

$$\Phi_{\ell,\sigma}(U,0^+) \stackrel{def}{=} \lim_{r \to 0^+} \Phi_{\ell,\sigma}(U,r).$$

The Almgren-Poon type monotonicity formula plays a crucial role in the blowup analysis of a solution to the problem.

We define the parabolic Almgren rescalings of U as

$$U_r = \frac{U \circ \delta_r}{H(U, r)^{1/2}}.$$

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We fix $\sigma \in (\frac{1-a}{2}, 1)$ and let $\kappa = \Phi_{2,\sigma}(U, 0+)$ (note $\ell = 2$). Then:

- There exists a sequence $r_j \to 0$ and a function $U_0 \in \mathbb{S}_\infty^+$ such that

$$\int_{\mathbb{S}_R^+} y^a ((U_{r_j} - U_0)^2 + |t||\nabla U_{r_j} - \nabla U_0|^2)\overline{\mathcal{G}}_a \to 0$$

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- U_0 is parabolically homogeneous of degree κ and is a global solution of the thin obstacle problem, i.e.,

$$\begin{cases} \mathcal{L}_a U_0 = 0 \quad \text{in } \mathbb{S}^+_\infty \\ U_0 \geq 0, \ \lim\limits_{y \to 0^+} y^a \partial_y U_0 \leq 0, \ U_0 \left(\lim\limits_{y \to 0^+} y^a \partial_y U_0 \right) = 0. \end{cases} \tag{0.1}$$

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Let $\sigma \in (0,1)$, $\ell \geq 4$ and $\kappa = \Phi_{\ell,\sigma}(U, 0+)$ be such that $\kappa < \ell - 1 + \sigma$. Then

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Definition 2

The set $\Lambda_\psi(u) = \{x \in \mathbb{R}^n : u(x) = 0\}$ is the coincidence set, and its boundary $\Gamma_\psi(u) = \partial\Lambda_\psi(u)$ is the free boundary.

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An important consequence of the gap theorem is:

The set of free boundary points which have minimal frequency $\kappa = \frac{3-a}{2}$ *is a relatively open subset of the free boundary.*

The regular free boundary

Definition 3

We define the regular part of the free boundary as the collection of all free boundary points $(X_0, t_0) = (x_0, 0, t_0)$ at which

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Theorem 4

The regular free boundary is locally a $\mathbb{H}^{\alpha,\alpha/2}$ *hypersurface.*

Proof: Reduction to elliptic case.

The singular set

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$$\lim_{r\to 0+} \frac{\mathcal{H}^{n+1}(\Lambda(v)\cap Q_r(X_0))}{\mathcal{H}^{n+1}(Q_r(X_0))} = 0.$$

We denote the set of singular points by $\Sigma(v)$ and call it the singular set. We can further classify singular points according to the homogeneity of their blowup, by defining

$$\Sigma_\kappa(v) := \Sigma(v)\cap\Gamma_\kappa^{(\ell)}(v), \quad \kappa\le\ell.$$

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Important fact: X_0 is singular $\Leftrightarrow$ $\kappa = 2m$, $m \in \mathbb{N}$.

Weiss type monotonicity formula in Gaussian space

To study the singular set we first prove the following

Theorem 5

Suppose that $\ell \geq 2$ is such that for some constant $C_\ell > 0$ one has $|F(X,t)| \leq C_\ell |(X,t)|^{\ell-2}$ for every $(X,t) \in \mathbb{S}_1^+$.
For $\kappa \in (0,\ell)$ we define the parabolic κ-Weiss type functional

$$\mathcal{W}_\kappa(U,r) \overset{def}{=} r^{-2\kappa}\Big\{D(U,r) - \frac{\kappa}{2}H(U,r)\Big\}.$$

Then, for any $0 < \sigma \leq \ell - \kappa$ there exists $C > 0$ depending on n, a, ℓ, C_ℓ such that the function $r \longrightarrow \mathcal{W}_\kappa(U,r) + Cr^{2\sigma}$ is monotonically nondecreasing in $(0,1)$, and therefore the limit

$$\mathcal{W}_\kappa(U,0^+) \overset{def}{=} \lim_{r\to 0^+} \mathcal{W}_\kappa(U,r)$$

exists.

Monneau type monotonicity formula in Gaussian space

A direct consequence of the Weiss monotonicity formula is the main tool to analyze singular points.

Theorem 6

Assume that for some $\ell \geq 3$ the function F satisfies the bounds $|F(X,t)| \leq C_\ell |(X,t)|^{\ell-2}$ in $\mathbb{S}_1^+$, $|\nabla F(X,t)| \leq C_\ell^\star |(X,t)|^{\ell-3}$ in $\mathbb{Q}_{1/2}^+$. Suppose that $0 \in \Sigma_\kappa(U)$ with $\kappa = 2m < \ell$, for $m \in \mathbb{N}$. For any parabolically κ-homogeneous polynomial p_κ in $\mathbb{S}_\infty$ we define the Monneau type functional

$$\mathcal{M}_\kappa \overset{def}{=} \frac{1}{r^{2\kappa+2}} \int_{\mathbb{S}_r^+} (U - p_\kappa)^2 \, \overline{\mathcal{G}}_a y^a, \qquad r \in (0,1).$$

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Then, for any $0 < \sigma < \ell - \kappa$ there exists a constant $C > 0$, depending on $n, a, \ell, C_\ell, \sigma$, such that the function $r \to \mathcal{M}_\kappa + C r^\sigma$ is monotonically nondecreasing on $(0,1)$.

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We show that at a singular point of homogeneity $\kappa = 2m$ such homogeneous blowup must be a parabolically κ-homogeneous polynomial p_κ satisfying

$$\mathcal{L}_a p_\kappa = 0, \quad p_\kappa(x,0,t) \geq 0, \quad p_\kappa(x,-y,t) = p_\kappa(x,y,t).$$

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Monneau monotonicity formula also implies another important piece of information: *The continuous dependence of the blowup from the free boundary points.*

Structure of the singular set

Combining these results with a parabolic Whitney type extension theorem we are able to establish the rectifiable structure of the singular set

Theorem 7

Let $F \in H^{\ell,\ell/2}(Q_1)$, $\ell \geq 3$. Then, for any $\kappa = 2m < \ell$, $m \in \mathbb{N}$, we have $\Gamma_\kappa(U) = \Sigma_\kappa(U)$.

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Theorem 7

Let $F \in H^{\ell,\ell/2}(Q_1)$, $\ell \geq 3$. Then, for any $\kappa = 2m < \ell$, $m \in \mathbb{N}$, we have $\Gamma_\kappa(U) = \Sigma_\kappa(U)$. Moreover, for every $d = 0, 1, \ldots, n-2$, the set $\Sigma_\kappa^d(U)$ is contained in a countable union of $(d+1)$-dimensional space-like $C^{1,0}$ manifolds and $\Sigma_\kappa^{n-1}(v)$ is contained in a countable union of $(n-1)$-dimensional time-like C^1 manifolds.

Thank you for your attention!