Regularity of optimal transport between planar convex domains

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Motivation and applications.

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Yes, in 2D.

Theorem (Savin-Y.)

For convex Ω_1 and Ω_2 in $\mathbb{R}^2$, and $p \in (0, +\infty)$,

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Estimate has to depend on the inner/ outer radii of the domains.

Some ingredients:
Assume $0 \in \partial\Omega_1 \cap \partial\Omega_2$, $u(0) = 0$ and $\nabla u(0) = 0$.

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John's lemma: There is an ellipse E_h such that $E_h \subset S_h \subset C_d E_h$.
The more E_h looks like $B_{\sqrt{h}}$, the better estimate we have for u at 0.

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Proposition

1) $|S_h| \sim h$, $|S_h \cap \Omega_1| \sim h$, $|\nabla u(S_h) \cap \Omega_2| \sim h$.

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1) $|S_h| \sim h$, $|S_h \cap \Omega_1| \sim h$, $|\nabla u(S_h) \cap \Omega_2| \sim h$.
2) If $S_h \sim E_h$, then $\nabla u(S_h) \sim E_h^{\perp}$.

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Lemma (Obliqueness)

The left tangents of Ω_1 and Ω_2 form an acute angle.
The right tangents....

Thank you!