The fractional unstable obstacle problem

Mariana Smit Vega Garcia
Joint work with Mark Allen

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University of Connecticut Hartford

Mariana.SmitVegaGarcia@wwu.edu

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In this talk I will overview the fractional unstable obstacle problem.

Connections to other problems

Classical obstacle problem

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Allen & SVG: fractional unstable obstacle problem

$$(-\Delta)^s u = \chi_{\{u>c\}}, \qquad 0 < s < 1.$$

The action of $(-\Delta)^s$ on $\psi \in C_0^2(\mathbb{R}^n)$ is given by

$$(-\Delta)^s \psi(x) = c_{n,s} \text{ p.v.} \int_{\mathbb{R}^n} \frac{\psi(x) - \psi(y)}{|x-y|^{n+2s}}\, dy,$$

understood in the sense of the principal value.

Connections to other problems

Two-phase obstacle problem: $\Delta u = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}}$.
Solutions can be obtained by minimizing the energy

$$J(u) = \int_\Omega (|\nabla u|^2 + 2(\lambda_+ u^+ + \lambda_- u^-))dx.$$

See Shahgholian-Uraltseva-Weiss (2004, 2007), Shahgholian-Weiss (2006), Uraltseva (2001), Weiss (2001).

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Two-phase fractional obstacle problem (Allen, Lindgren & Petrosyan, 2015): Study minimizers of the energy

$$\int_{\Omega^+} |\nabla u|^2 x_n^a + 2\int_{\Omega'} (\lambda_+ u^+ + \lambda_- u^-),$$

where $\Omega \subset \mathbb{R}^n$, $\Omega^+ = \Omega \cap \{x_n > 0\}$, $\Omega' = \Omega \cap \{x_n = 0\}, \lambda_\pm > 0$.

Fractional unstable obstacle problem

Allen & SVG, 2018:

Study minimizers of the energy

$$J_a(u) = \int_{\Omega^+} |\nabla u|^2 x_n^a - 2\int_{\Omega'} (\lambda_+ u^+ + \lambda_- u^-) d\mathcal{H}^{n-1}.$$

where $\Omega \subset \mathbb{R}^n$, $\Omega^+ = \Omega \cap \{x_n > 0\}$, $\Omega' = \Omega \cap \{x_n = 0\}, \lambda_\pm \geq 0$.

Minimization occurs over $H^1(a, \Omega^+)$ with fixed boundary data on $\partial\Omega \cap \{x_n > 0\}$.

Temperature control through the boundary

When $s = 1/2$, solutions of

$$(-\Delta)^s u = \chi_{\{u>c\}}, \qquad 0 < s < 1$$

model temperature control on the boundary (see Duvaut-Lions, Inequalities in Mechanics and Physics).

More heat is injected when the temperature rises on the boundary - corresponding to a boundary reaction.

Localization of the fractional Laplacian

$U \subset \mathbb{R}^{n-1}$ bounded, $(x', x_n) \in \mathbb{R}^n$.

Let $u(x')$ solve $(-\Delta)^s u = \chi_{\{u>c\}}$. Extend u to $U \times \mathbb{R}$ adding variable x_n:

$$\begin{aligned} &\operatorname{div}(x_n^a \nabla u(x', x_n)) = 0 \text{ in } U \times \mathbb{R}^+; \\ &\lim_{x_n \to 0} x_n^a \partial_{x_n} u(x', x_n) = \frac{1}{c_{n,a}} \chi_{\{u(x',0)>c\}}, \end{aligned} \tag{0.2}$$

where $2s = 1 - a$.

Solutions to (0.2) can be found by minimizing

$$\int_{U \times \mathbb{R}^+} |\nabla v(x', x_n)|^2 x_n^a - \frac{2}{-c_{n,a}} \int_U (v - c)^+ d\mathcal{H}^{n-1}.$$

Generalized solution

$\Omega \subset \mathbb{R}^n$: bounded smooth domain, even in x_n,
$(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$,
$\Omega' = \Omega \cap \{x_n = 0\}$,
$\Omega^+ = \Omega \cap \{x_n > 0\}$,
$\lambda_\pm \geq 0$.
Goal: minimize

$$J_a(v, \lambda_+, \lambda_-) = \int_{\Omega^+} |\nabla v|^2 |x_n|^a - 2 \int_{\Omega'} (\lambda_+ v^+ + \lambda_- v^-) d\mathcal{H}^{n-1}, \tag{0.3}$$

over $\{v \in H^1(a, \Omega^+) \ : \ v = \varphi \text{ on } \partial\Omega \cap \{x_n > 0\}\}$.

Minimizing (0.3) is always a "two-phase" problem

Proposition

u is a minimizer of $J_a(v, \lambda_+, \lambda_-) \iff u + cx_n^{1-a}$ is a minimizer of

$$J_a(w, \lambda_+ - c(1-a), \lambda_- + c(1-a)),$$

for any c with $-\lambda_- \le c(1-a) \le \lambda_+$.

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for any c with $-\lambda_- \leq c(1-a) \leq \lambda_+$.

Consequently: focus on minimizing

$$J_a(v) = \int_{\Omega^+} |\nabla v|^2 |x_n|^a - 2\int_{\Omega'} v^-.$$

Equations

Minimizers of J_a solve

$$\int_{\Omega^+} x_n^a \langle \nabla u, \nabla \psi \rangle = \int_{\Omega' \cap \{u<0\}} -\psi, \qquad \forall\ \psi \in C_0^1(\Omega).$$

Moreover,

$$\operatorname{div}(x_n^a \nabla u(x', x_n)) = 0 \text{ in } \Omega^+;$$
$$\lim_{x_n \to 0} x_n^a \partial_{x_n} u(x', x_n) = \chi_{\{u(x',0)<0\}}.$$

Main goals

- Existence of minimizers;
- Regularity of minimizers;
- Topological properties of the free boundary;
- Upper bound for the Hausdorff dimension of the singular set of the free boundary.

Existence of minimizers to J_a

- Compactness of trace operators (Allen, Lindgren, Petrosyan, 2015);
- Boundedness of J_a from below;
- Convexity and closedness of set of candidates;
- Standard methods of calculus of variations.

Initial properties

Nondegeneracy in the full domain Ω^+:

If u minimizes J_a in $B_R(x_0, 0)^+$ and $u(x_0, 0) = 0$, then

$$\sup_{B_r(x_0,0)^+} u \geq C r^{1-a}, \qquad \forall\, r < R.$$

A non-degeneracy result for the thin space Ω' is obtained later.

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Weighted boundary derivative on Ω' for u is constant in a measure theoretic sense when $u < 0$:

Let u minimize J_a in Ω. Then $\forall \psi \in C_0^2(\Omega)$,

$$\int_{\Omega^+} x_n^a \langle \nabla u, \nabla \psi \rangle = \int_{\Omega' \cap \{u<0\}} -\psi.$$

Regularity for minimizers of J_a

Case $0 < s < \frac{1}{2}$: $u \in C^{0,1-a}(\Omega^+ \cup \Omega')$ and

$$||u||_{C^{0,1-a}\left(\overline{B^+_{r/2}}\right)} \leq C||u||_{L^2(a,B^+_r)}.$$

Allen-Lindgren-Petrosyan: (two-phase fractional obstable problem)
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Allen-Lindgren-Petrosyan: (two-phase fractional obstable problem) $u \in C^{0,1-a}\left(\overline{B^+_{r/2}}\right)$.

Case $\frac{1}{2} < s < 1$: $u \in C^{1,-a}(\Omega^+ \cup \Omega')$ and

$$||u||_{C^{1,-a}\left(\overline{B^+_{r/2}}\right)} \leq C||u||_{L^2(a,B^+_r)}.$$

Allen-Lindgren-Petrosyan: $u \in C^{1,-a}\left(\overline{B^+_{r/2}}\right)$.

Monneau-Weiss: (minimizers of the unstable obstacle problem) $u \in C^{1,1}_{\text{loc}}$.

Regularity for minimizers of J_a

Case $s = 1/2$: $u \in C^{0,\alpha}(B^+_{1/2} \cup B'_{1/2})$ for every $0 < \alpha < 1$ and

$$||u||_{C^{0,\alpha}\left(\overline{B^+_{1/2}}\right)} \leq C||u||_{L^2(B_1)}.$$

Allen-Lindgren-Petrosyan: (minimizers of the two-phase fractional obstable problem) $u \in C^{0,1}\left(\overline{B^+_{r/2}}\right)$.

Remark: In the fractional unstable obstacle problem, we may not expect Lipschitz regularity when $s = 1/2$.

Free boundary

$$\Gamma^+ = \partial\{u(\cdot,0) > 0\}, \qquad \Gamma^- = \partial\{u(\cdot,0) < 0\}, \qquad \Gamma = \Gamma^+ \cup \Gamma^-.$$

Case $s > 1/2$: minimizers are $C^{1,-a}$.
Implicit function theorem $\Rightarrow \Gamma$ is $C^{1,-a}$ manifold wherever $|\nabla u| \neq 0$.

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Implicit function theorem $\Rightarrow$ Γ is $C^{1,-a}$ manifold wherever $|\nabla u| \neq 0$.

Case $s \leq 1/2$: minimizers are $C^{0,1-a}$. Minimizers are not differentiable.
More complicated!

Free boundary

$$\Gamma^+ = \partial\{u(\cdot,0) > 0\}, \qquad \Gamma^- = \partial\{u(\cdot,0) < 0\}.$$

If u minimizes J_a, then

- $\{(x,0) \mid u(x,0) = 0\}$ has no interior point in the topology of $\mathbb{R}^{n-1}$,
- $\mathcal{H}^{n-1}(\Gamma^+ \cup \Gamma^-) = 0$.

This is what is expected, as it is true when $s = 1$.

Weiss-type monotonicity formula:

$$W(r) = \frac{1}{r^{n-a}} \int_{B_r^+} x_n^a |\nabla u|^2 - \frac{2}{r^{n-a}} \int_{B_r'} u^- - \frac{1-a}{r^{n+1-a}} \int_{(\partial B_r)^+} x_n^a u^2$$

is nondecreasing for $0 < r < 1$.
W is constant on $[r_1, r_2] \iff u$ is homogeneous of degree $2s = 1 - a$ on $r_1 < |x| < r_2$.

Allows us to show blow-ups are homogeneous of degree $2s$.

Convergence of rescalings

Let u be a minimizer of J_a with $u(x_0, 0) = 0$.

- If $a = 0$ assume

$$\sup_{B_r(x_0,0)} |u| \leq Cr \qquad \forall r < r_0 \text{ for some } r_0.$$

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- If $a < 0$, assume $\nabla_x u(x_0, 0) = 0$.

Then for any $r_k \to 0$,

$$u_{r_k}(x) = \frac{u(x_0 + r_k x)}{r_k^{2s}}$$

converges (up to subsequence) to u_0, minimizer of J_a in every $K \subset\subset \mathbb{R}^n$.
u_0 is homogeneous of degree $2s = 1 - a$.

Almgren's frequency formula

Let $u \in H^1(B_1)$ solve $\Delta u = 0$. Then

$$N(r) = r\frac{\int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2}$$

is nondecreasing for $r > 0$. Moreover, $N(r) \equiv \kappa$ if and only if u is homogeneous of degree κ.

Convergence of rescalings when $a = 0$

Fix $a = 0$ ($s = 1/2$). Let u minimize J_a with $u(x_0, 0) = 0$ and not satisfy

$$\sup_{B_r(x_0,0)} |u| \leq Cr, \; \forall \; r < r_0 \text{ for some } r_0.$$

Then

$$u_r(x) = \frac{u(rx + x_0)}{\left(r^{1-n} \int_{\partial B_r(x_0)} u^2\right)^{1/2}}$$

is bounded in $H^1(B_1)$ and every limit u_0 as $r \to 0$ is a linear function in the x' variable.

Consequence of Weiss + Almgren.

Classification

If u is a-harmonic in Ω^+, if it is homogeneous of degree $1-a$ and

$$\int_{\Omega^+} x_n^a \langle \nabla u, \nabla \psi \rangle = -c \int_{\Omega'} \psi, \qquad \forall \psi \in C_0^1(\Omega),$$

then $u \equiv c x_n^{1-a}/(1-a)$.

Comparison of phases

Allen, Lindgren, Petrosyan (2015):

Let $a \geq 0$. For minimizers of

$$\int_{\Omega^+} |\nabla v|^2 x_n^a + 2\int_{\Omega'} \lambda_+ u^+ + \lambda_- u^-,$$

$$\Gamma^+ \cap \Gamma^- = \emptyset.$$

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Let $a \geq 0$. For minimizers of

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$$\Gamma^+ \cap \Gamma^- = \emptyset.$$

Allen, SVG, 2018:

Let u be a minimizer of J_a. Then

$$\Gamma^+ = \Gamma^-.$$

Non-degeneracy for the thin space

Let u minimize J_a in B_R with $u(0,0) = 0$.

- If $a \neq 0$, then for $C = C(n,a)$,

$$\sup_{B'_r} u^+, \ \sup_{B'_r} u^- \geq Cr^{1-a}, \qquad \text{for } r < R/2.$$

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- If $a = 0$, then for $C = C(n, ||u||_{L^2(B_R)})$,

$$\sup_{B_r'} u^+, \ \sup_{B_r'} u^- \geq C \left(r^{1-n} \int_{\partial S_r} u^2 \right)^{1/2} \qquad \text{for } r < R/2.$$

SINGULAR POINTS

Singular points for $s > 1/2$

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$$S_u = \Gamma \cap \{\nabla u = 0\}.$$

Implicit function theorem $\Rightarrow$ $\Gamma \setminus \{\nabla u = 0\}$ is a $C^{1,-a}$ surface of co-dimension 2.

Classification of blow-ups, for $n = 2$

For $a \neq 0$:

There exists at most one not identically zero global minimizer of J_a homogeneous of degree $1 - a$.

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For $a = 0$ $(s = 1/2)$:

It had already been shown that blow-ups are linear functions.

Singular points for $s \leq 1/2$

Definition

For $s \leq 1/2$: at most one non-zero global minimizer of J_a homogeneous of degree $2s$ for $n = 2$, $g(x_1, x_2)$. Define

$$\check{g}(x_1, x_2, \ldots, x_n) = g(x_1, x_n).$$

S_u: set of $x \in \Gamma$ such that if u_0 is any blow-up of u at x, then u_0 is a rotation in the first $n-1$ variables of $\check{g}$.

Monneau & Weiss, 2007:

Let u be a minimizer of the unstable obstacle problem. The Hausdorff dimension of S_u is $\leq n-2$.

Allen, SVG, 2018:

Let $n=3$ and $s>1/2$, u minimizer of J_a in Ω.
For any $K \subset\subset \Omega$, $K \cap S_u$ contains at most finitely many points.

Allen, SVG, 2018:

Let u be a minimizer of J_a with $s>1/2$. The Hausdorff dimension of S_u is $\leq n-3$.

Ingredients: $C^{1,\alpha}$ convergence of scalings to blow-ups; non-degeneracy; blow-ups are homogeneous of degree $1-a$; classification of global minimizers homogeneous of degree $1-a$ when $n=2$; dimension reduction argument of Federer.

Thank you!