Discrete curvatures and geometry of measures

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Given three points $x, y, z \in \mathbb{R}^m$ define

$$c(x, y, z) = \frac{1}{R(x, y, z)}$$

where $R(x, y, z)$ is the radius of the unique circle containing x, y, z. If x, y, z are colinear, $c(x, y, z) = 0$.

A Borel measure μ on $\mathbb{R}^m$ is said to be countably n-rectifiable if $\mu \ll \mathcal{H}^n$ and there exist Lipschitz maps $f_i : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\mu\left(\mathbb{R}^m \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^n)\right) = 0.$$

A set $E \subset \mathbb{R}^m$ is said to be countably n-rectifiable if $\mathcal{H}^n \llcorner E$ is countably n-rectifiable.

Theorem (Leger '99)

If μ is a 1-Ahlfors regular Borel measure on $\mathbb{R}^m$, then μ is countably 1-rectifiable if and only if μ has σ-finite integral Menger curvature in the sense that there are countably many sets F_j such that $\mu_j = \mu \llcorner F_j$ satisfy

$$\iiint c^2(x,y,z)\,d\mu_j^3(x,y,z) < \infty \qquad \forall j$$

Corollary

Let E be a compact 1-Ahlfors regular subset of $\mathbb{C}$. The following three are equivalent:

(i) *E is removable for bounded analytic functions*

(ii) *E is removable for Lipschitz harmonic functions*

(iii) *E is purely unrectifiable*

Corollary

Let E be a compact 1-Ahlfors regular subset of $\mathbb{C}$. The following three are equivalent:

(i) *E is removable for bounded analytic functions*

(ii) *E is removable for Lipschitz harmonic functions*

(iii) *E is purely unrectifiable*

Theorem (Long history, Mattila, Melnikov, Verdera '96)

Let E be a closed 1-Ahlfors regular subset of $\mathbb{C}$. Then, the following are equivalent

(a) $\displaystyle\int_{B\cap E}\int_{B\cap E}\int_{B\cap E} c(x,y,z)^2 d\mathcal{H}^1(x)d\mathcal{H}^1(y)d\mathcal{H}^1(z) \leq C\,\mathrm{diam}(B),\ \forall B.$

(b) $\int_E \left|\int_{E\setminus B(x,\epsilon)} \frac{x-y}{|x-y|^2} g(y)d\mathcal{H}^1(y)\right|^2 d\mathcal{H}^1(x) \leq C\int_E |g|^2 d\mathcal{H}^1$ *for all $g \in L^2(E)$ for all $\epsilon > 0$, and some C independent of ϵ.*

(c) *E is contained in an Ahlfors upper-regular curve Γ.*

How?

An identity due to Melnikov in 1995 shows

$$c^2(z_1, z_2, z_3) = \sum_{\sigma \in S_3} \frac{1}{\left(z_{\sigma(1)} - z_{\sigma(3)}\right) \overline{\left(z_{\sigma(2)} - z_{\sigma(3)}\right)}}$$

which directly relates Menger curvature to the L^2-boundedness of the Cauchy kernel in $\mathbb{C}$. This can be seen roughly because

$$\int \left| \int \frac{d\mu(\zeta)}{\zeta - z} \right|^2 d\mu(z) = \int \left\{ \int \frac{d\mu(\zeta)}{(\zeta - z)} \int \frac{d\mu(\xi)}{\left(\overline{\xi - z}\right)} \right\} d\mu(z)$$

$$= \iiint \frac{1}{(\zeta - z)\left(\overline{\xi - z}\right)} d\mu(\zeta) d\mu(\xi) d\mu(z).$$

Theorem (Farag, 2000)

"Curvatures" who look remotely like any reasonable higher-dimensional generalization of

$$\sum_{\sigma} \frac{1}{\left(z_{\sigma(1)} - z_{\sigma(3)}\right)\left(\overline{z_{\sigma(2)} - z_{\sigma(3)}}\right)}$$

fail to be non-negative.

The Jones β-numbers detect how far a set or measure is from being (contained in) a plane at a given location and scale.

$$\beta^n_{\mu;2}(x,r) = \inf_L \left(\frac{1}{r^n} \int_{B(x,r)} \left(\frac{\operatorname{dist}(y,L)}{r} \right)^2 d\mu(y) \right)^{\frac{1}{2}},$$

where the infimum is taken over all n-planes.

Theorem (Tolsa, Azzam and Tolsa, 2015)

Let μ be a Radon measure in $\mathbb{R}^m$ such that $0 < \limsup_{r\to 0} \frac{\mu(B(x,r))}{r^n} < \infty$ for μ almost every $x \in \mathbb{R}^m$. Then, μ is countably n-rectifiable if and only if

$$\int_0^1 \beta_{\mu;2}^n(x,r)^2 \frac{dr}{r} < \infty \qquad \mu - a.e.\ x \in \mathbb{R}^m$$

Classical Menger curvature and rectifiability

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Generalizing Menger curvature

1. For $x_0, \ldots, x_{n+1} \in \mathbb{R}^m$ let

$$\mathcal{K}^2(x_0, \ldots, x_{n+1}) = \frac{h_{\min}(x_0, \ldots, x_{n+1})^2}{\operatorname{diam}\{x_0, \ldots, x_{n+1}\}^{n(n+1)+2}}$$

where $h_{\min}(x_0, \ldots, x_{n+1}) = \min\{\operatorname{dist}(x_i, \operatorname{aff}\{x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}\})\}$

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2. Compare

$$c(x_0, x_1, x_2)^2 = \frac{4\sin^2(\alpha_0)}{|x_1 - x_2|^2} = \frac{4\sin^2(\alpha_0)|x_0 - x_2|^2}{|x_1 - x_2|^2|x_0 - x_2|^2}$$

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3. Question: Why not define MC in terms of the radius of the circumsphere containing the $(n+2)$-points?

We define the integral Menger curvature of μ by

$$\mathcal{M}_{\mathcal{K}}(\mu) = \int_{(\mathbb{R}^m)^{n+2}} \mathcal{K}^2(x_0, \ldots, x_{n+1}) d\mu^{n+2}(x_0, \ldots, x_{n+1})$$

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We define the pointwise Menger curvature of μ at x and scale r by

$$\mathrm{curv}_\mu(x, r) = \int_{B(x,r)^{n+1}} \mathcal{K}^2(x, x_1, \ldots, x_{n+1}) d\mu^{n+1}(x_1, \ldots, x_{n+1})$$

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Notably,

$$\frac{\mathcal{M}_{\mathcal{K}}(\mu \llcorner B(x, r))}{r^n} = \mathcal{M}_{\mathcal{K}}(\mu_{x,r} \llcorner B(0, 1)) \text{ and } \mathrm{curv}_{\mu}(x, r) = \mathrm{curv}_{\mu_{x,r}}(0, 1)$$

where

$$\mu_{x,r}(A) = \mu(rA + x)$$

A sufficient condition for rectifiable sets

Theorem (Meurer '15)

If E is a Borel set on $\mathbb{R}^m$ and

$$\mathcal{M}_{\mathcal{K}^2}(\mathcal{H}^n \llcorner E) < \infty,$$

then E is countably n-rectifiable.
(Also, $\mathcal{H}^n(E \cap K) < \infty$ for all compact sets K.)

Menger curvature and β-numbers

Theorem (Kolasinski '16)

Let μ be a Borel measure on $\mathbb{R}^m$ with some density conditions. Then,

$$\operatorname{curv}_\mu(x, R) \lesssim \int_0^{\Gamma R} \hat{\beta}^n_{\mu;2}(x, r)^2 \frac{dr}{r}$$

for some dimensional constant Γ. In particular, if μ is countably n-rectifiable, then

$$\operatorname{curv}_\mu(x, 1) < \infty \qquad \mu \text{ a.e. } x \in \mathbb{R}^m$$

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Lemma (Kolasinski, 2016)

If μ satisfifes $0 < \Theta^n_*(\mu, x) \leq \Theta^{n,*}(\mu, x) < \infty$, the following are equivalent:

1. $\int_0^1 \beta^n_{\mu;2}(x, r)^2 \frac{dr}{r} < \infty$ *for μ almost every x*
2. $\int_0^1 \hat{\beta}^n_{\mu;2}(x, r)^2 \frac{dr}{r} < \infty$ *for μ almost every x.*

Notably,

$$\frac{\mathcal{M}_{\mathcal{K}}(\mu \llcorner B(x,r))}{r^n} = \mathcal{M}_{\mathcal{K}}(\mu_{x,r} \llcorner B(0,1)) \text{ and } \operatorname{curv}_\mu(x,r) = \operatorname{curv}_{\mu_{x,r}}(0,1)$$

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Question 1: Can you characterize rectifiable sets in terms of $\lim_{r\downarrow 0} \frac{\mathcal{M}_{\mathcal{K}}(\mu \llcorner B(x,r))}{r^n}$?

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Question 2: Can you extend Meurer's theorem to some class of measures?
Question 3: Can you characterize rectifiable measures with MC in any form?

Answer 1: Failure of density of $\mathcal{M}_{\mathcal{K}}(\mu \llcorner B(x, r))$

Theorem (G. and McCurdy 2019)

There exists a 1*-Ahlfors regular set* $A_0 \subset [0,1]^2$ *such that* $\mu = \mathcal{H}^1 \llcorner A_0$ *satisfies* $0 < \mu(\mathbb{R}^2) < \infty$ *and* μ *is countably* 1*-rectifiable, but*

$$\mathcal{M}_{\mathcal{K}}(\mu \llcorner B(x, \delta)) \equiv +\infty \qquad \forall x \in A_0, \quad \forall \delta > 0$$

Answers 2 and 3

Theorem (G. '17)

If μ is a Radon measure with $0 < \Theta^{*,n}(\mu, x) < \infty$ for μ a.e. $x \in \mathbb{R}^m$ and $\mathcal{M}_{\mathcal{K}}(\mu) < \infty$ then μ is countably n-rectifiable.

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Theorem (G. '17)

Let μ be a Radon measure with $0 < \Theta_*^n(\mu, x) \leq \Theta^{*,n}(\mu, x) < \infty$ for μ a.e. $x \in \mathbb{R}^m$ then the following are equivalent:

1) *μ is countably n-rectifiable.*
2) *For μ almost every $x \in \mathbb{R}^m, \mathrm{curv}_\mu^n(x, 1) < \infty$.*
3) *μ has σ-finite integral Menger curvature in the sense that μ can be written as*

$$\mu = \sum_{j=1}^{\infty} \mu_j \qquad \text{where} \qquad \mathcal{M}_{\mathcal{K}}(\mu_j) < \infty \quad \forall j.$$

Theorem (Kolasinski 2017, Ghinassi, G. 2018)

Let

$$\mathrm{curv}^{\alpha}_{\mu;2}(x,r) = \int_{(B(x,r))^{n+1}} \frac{h_{\min}(x, x_1, \ldots, x_{n+1})^2}{\mathrm{diam}\left(\{x, x_1, \ldots, x_{n+1}\}\right)^{n(n+1)+2+2\alpha}} d\mu^{n+1}(x_1, \ldots, x_{n+1}).$$

Let μ be a Radon measure on $\mathbb{R}^m$ such that $0 < \Theta^{n,*}(\mu, x) < \infty$ for μ-almost every $x \in \mathbb{R}^m$ and $\alpha \in [0,1)$. Moreover, suppose that for μ- a.e. $x \in \mathbb{R}^m$

$$\mathrm{curv}^{\alpha}_{\mu;2}(x,1) < \infty.$$

Then μ is countably $(n, C^{1,\alpha})$-rectifiable.

Thank you!