Strong-form stability for the Sobolev inequality

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Regularity Theory of PDEs and Calculus of Variations on Domains with Rough Boundaries

The Euclidean Sobolev inequality

A variational problem

Fix $n \geq 2$ and $p \in [1, n)$, and set $p^* = \frac{np}{n-p}$.

Consider the minimization problem

$$\inf \left\{ \|\nabla u\|_p \ : \ \|u\|_{p^*} = 1, \ u \in C_0^\infty(\mathbb{R}^n) \right\} := S \underbrace{> 0}_{\text{Sobolev}} .$$

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The Euclidean Sobolev inequality

In other words, we have the Sobolev inequality

$$\|\nabla u\|_p \geq S\|u\|_{p^*}$$

for all $u \in W^{1,p}(\mathbb{R}^n)$.

Extremal functions

For $p > 1$, equality is achieved, i.e. $\|\nabla v\|_p = S\|v\|_{p^*}$ for

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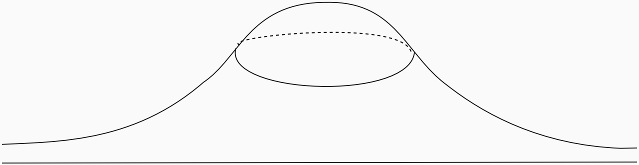

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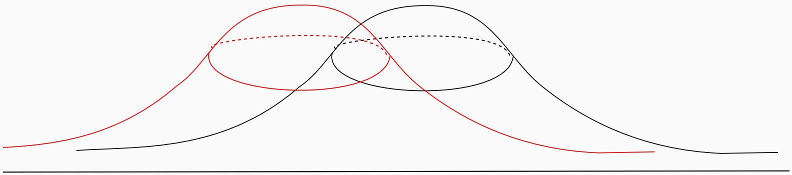

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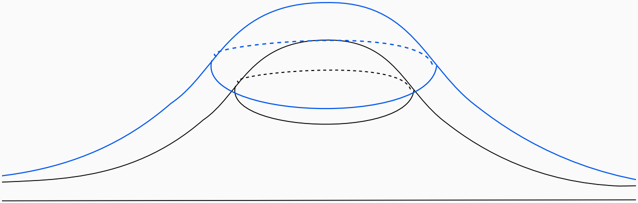

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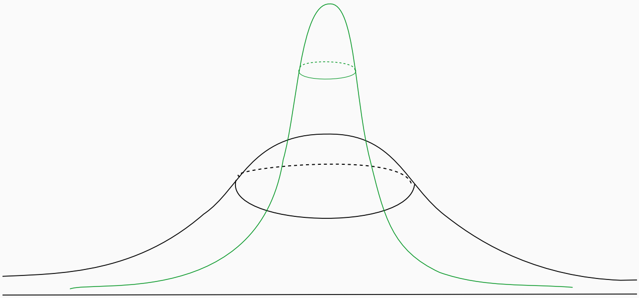

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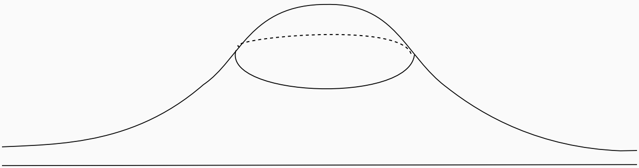

All equality cases are

$$\mathcal{M} = \{cv(\lambda(x - x_0)) : c \in \mathbb{R}, \lambda \in \mathbb{R}_+, x_0 \in \mathbb{R}^n\}.$$

Quantitative Stability: If $u \in W^{1,p}$ *almost* attains equality in the Sobolev inequality, then how close is u to an extremal function $v \in \mathcal{M}$?

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Normalize so $\|u\|_{p^*} = 1$ and define the deficit

$$\delta(u) = \begin{cases} \|\nabla u\|_p^p - S^p \|u\|_{p^*}^p & \text{if } p \in [2, n) \\ \|\nabla u\|_p^{p'} - S^{p'} \|u\|_{p^*}^{p'} & \text{if } p \in (1, 2) \end{cases}$$

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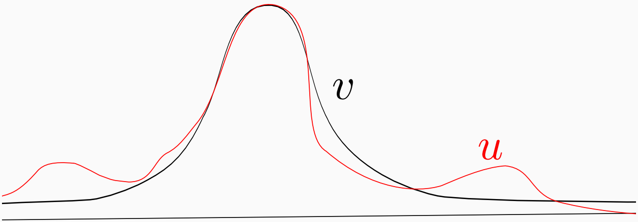
The strongest distance we expect to control in this setting is

$$d(u, \mathcal{M}) = \inf_{v \in \mathcal{M}} \|\nabla u - \nabla v\|_p.$$



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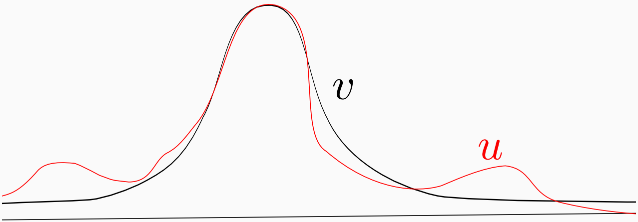

Quantitative stability for the Sobolev inequality

Want to prove an estimate of the type

$$\delta(u) \geq \omega\left(d(u, \mathcal{M})\right)$$

Expected optimal result:

$$\omega(t) = t^{\max\{2,p\}}$$

Stability in the sense of second variation

A completely unjustified computation

Given a function $u \in W^{1,p}$, let $v \in \mathcal{M}$ be such that

$$d(u, \mathcal{M}) = \|\nabla u - \nabla v\|_p.$$

A Taylor expansion of $\delta(u)$ at v gives

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$$\delta(u) = \delta(v) + \text{first variation} + \frac{1}{2}\text{second variation} + \text{h.o.t.}$$

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$$\delta(u) = \frac{1}{2}\text{second variation} + \text{h.o.t.}$$

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So does this work?

If $p \geq 2$, then a spectral analysis shows that

$$\text{second variation} \geq c \int |\nabla v|^{p-2} |\nabla u - \nabla v|^2.$$

($p = 2$ Bianchi-Egnell, $p > 2$ Figalli, N.)

If $p = 2$, yes

If $p = 2$, then this shows that $\delta(u) \geq cd(u, \mathcal{M})^2$.
(Bianchi-Egnell)

If $p > 2$, then kind of

If $p > 2$, then this strategy combined with an interpolation argument shows that $\delta(u) \geq cd(u, \mathcal{M})^{\alpha}$.

(Figalli, N.)

If $p \in (1, 2)$, then absolutely not

If $p \in (1, 2)$, we cannot cannot write down the second variation, the function $t \mapsto t^p$ is not twice differentiable at $t = 0$.

Quantitative stability in terms of the L^{p^*} norm

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Theorem (Cianchi, Fusco, Maggi, Pratelli '07)
For $p \in (1, n)$ and $u \in W^{1,p}$, we have

$$\delta(u) \geq c \inf_{v \in \mathcal{M}} \|u - v\|_{p^*}^{\beta}.$$

Optimal transport and symmetrization techniques. Control of gradients seems out of reach with this approach.

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Main result

A general reduction theorem

Theorem (N. '19)

For any $p \in (1,n)$ *and* $u \in W^{1,p}$, *and* $v \in \mathcal{M}$ *with* $\|u\|_{p^*} = \|v\|_{p^*} = 1$, *we have*

$$\|\nabla u - \nabla v\|_p^\alpha \leq C_1\,\delta(u) + C_2\|u - v\|_{p^*}\,.$$

Here, $\alpha = p'$ *if* $p \in (1,2)$ *and* $\alpha = p$ *if* $p \in [2,n)$.

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Corollary: strong-form quantitative stability

Corollary (N. '19)

For all $p \in (1, n)$ *and* $u \in W^{1,p}(\mathbb{R}^n)$*, we have*

$$\delta(u) \geq c d(u, \mathcal{M})^{\beta'} .$$

A bit on the proof

In five words

Proof.
Convexity and the Sobolev inequality. $\square$

Clarkson's inequalities

Let $F, G : \mathbb{R}^n \to \mathbb{R}^n$ with $|F|, |G| \in L^p(\mathbb{R}^n)$. Then

$$\left\|\frac{F+G}{2}\right\|_p^{p'} + \left\|\frac{F-G}{2}\right\|_p^{p'} \leq \left(\frac{1}{2}\|F\|_p^p + \frac{1}{2}\|G\|_p^p\right)^{p'/p}$$

if $p \in (1,2)$, and

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Proof sketch, $p \in (1,2)$

Take $p \in (1,2)$, normalize so $\|u\|_{p^*} = 1$.

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\begin{aligned}
& \left\| \frac{\nabla u - \nabla v}{2} \right\|_p^{p'} \\
\text{Clarkson's inequality} \quad & \leq \left(\frac{1}{2} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla v\|_p^p \right)^{p'/p} - \left\| \frac{\nabla u + \nabla v}{2} \right\|_p^{p'} \\
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The end

Thank you for your attention!