UCONN
UNIVERSITY OF CONNECTICUT

**Fall 2018 - Math 2410** **Name (Print): Solution KEY**
**Exam 1 - September 27**
**Time Limit: 75 Minutes**

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This exam contains 10 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

| Problem | Points | Score |
|---|---|---|
| 1 | 18 | |
| 2 | 10 | |
| 3 | 12 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 20 | |
| 8 | 10 | |
| Total: | 100 | |

1. Consider the following differential equations.

   A. $x\frac{dy}{dx} + 6ye^y = y^2$.
   B. $y'' + \sin(x)\, y' + \cos(x)\, y = \tan(x)$.
   C. $2\frac{dy}{dx} - y = e^x$.
   D. $2xy^5y^{(4)} + 2y' - e^x y = \cos(x)$.

   (a) (4 points) Write down the order of the differential equations in (A) to (D).
   A. has order **1**.
   B. has order **2**.
   C. has order **1**.
   D. has order **4**.

   (b) (4 points) Test the differential equations from (A) to (D) if they are linear or nonlinear.
   A. is a **nonlinear**.
   B. is a **linear**.
   C. is a **linear**.
   D. is a **nonlinear**.

   (c) (3 points) Which of the differential equations in (A)-(D) is a first order separable equation? If there is no such equation, write NONE.

   **Solution:** Only A. is separable.

   (d) (3 points) Which of the differential equations in (A)-(D) is first order autonomous differential equation? If there is no such equation, write NONE.

   **Solution:** NONE.

   (e) (4 points) Which of the differential equations in (A)-(D) is a linear and first order? Find an integrating factor that could be used to solve that differential equaion.

   **Solution:** C. is the only linear first order equation. Since it is the linear and first order and if rewrite the DE, we have

$$\frac{dy}{dx} - \frac{1}{2}y = \frac{1}{2}e^x.$$

   Then the integrating factor is

$$\mu(x) = e^{-\int \frac{1}{2}dx} = e^{-\frac{x}{2}}.$$

2. Consider the following differential equation

$$(\sin(y) - y\sin(x))dx + (\cos(x) + x\cos(y) - y)dy = 0$$

(a) (3 points) Show that the differential equation is exact.

**Solution:** Here $M(x,y) = \sin(y) - y\sin(x)$ and $N(x,y) = \cos(x) + x\cos(y) - y$. To check exactness we need to check if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Now since

$$\frac{\partial M}{\partial y} = \cos(y) - \sin(x)$$

and

$$\frac{\partial N}{\partial x} = -\sin(x) + \cos(y)$$

we see that the differential equation is exact.

(b) (4 points) Find the 1-parameter family of solution of the differential equation (leave the solution as an implicit function).

**Solution:** Since the differential equation is exact, we can find $f(x,y) = 0$ for which

$$\frac{\partial f}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x,y).$$

We can find $f$ by integrating

$$f(x,y) = \int \frac{\partial f}{\partial x} dx = \int M(x,y)dx = \int (\sin(y) - y\sin(x))dx = x\sin(y) + y\cos(x) + g(y)$$

where $g$ is a (differentable) function of $y$. Since we also know that

$$\frac{\partial f}{\partial y} = N(x,y) = \cos(x) + x\cos(y) - y$$

From above we also know that

$$\frac{\partial f}{\partial y} = \frac{x\sin(y) + y\cos(x) + g(y)}{\partial y} = x\cos(y) + \cos(x) + g'(y).$$

Hence combining these two identities we get

$$\cos(x) + x\cos(y) - y = x\cos(y) + \cos(x) + g'(y)$$

which gives us $g'(y) = -y$ and therefore $g(y) = -y^2/2 + c$. We conclude that

$$f(x,y) = x\sin(y) + y\cos(x) - \frac{y^2}{2} + c = 0$$

is the implicit solution.

(c) (3 points) Find the particular solution to the initial value problem $y(0) = 1$.

**Solution:** Since $x\sin(y) + y\cos(x) - \frac{y^2}{2} + c = 0$ is the general solution, we have

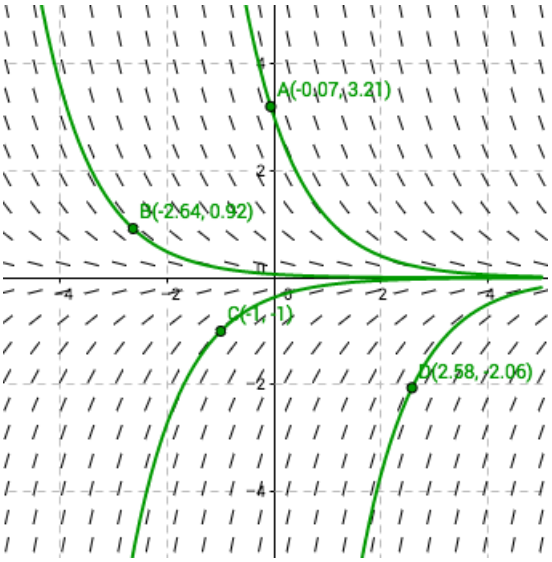
$$0 + 1 - \frac{1}{2} + c = 0 \quad c = -\frac{1}{2}.$$

Therefore

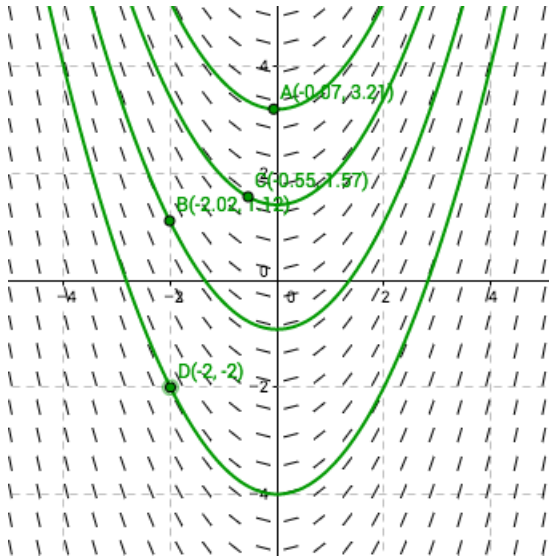
$$x\sin(y) + y\cos(x) - \frac{y^2}{2} = \frac{1}{2}$$

is the particular solution.

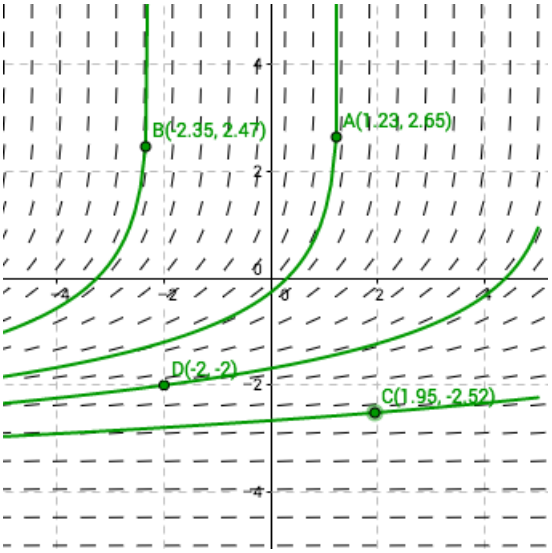
3. Consider the following direction fields(slope field).



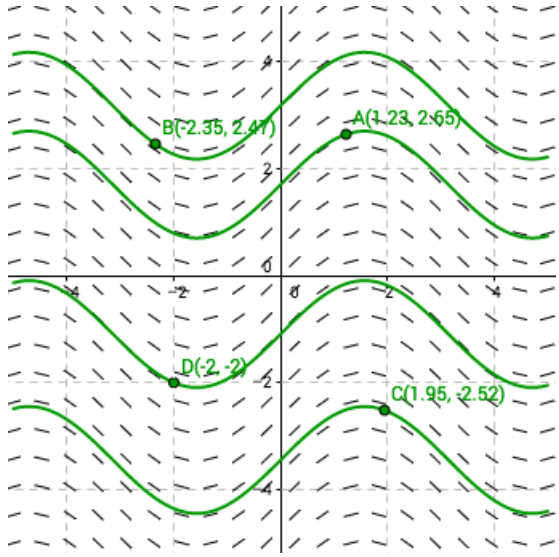

(a) ...




(b) ...




(c) ...




(d) ...

(a) (4 points) Match the given direction fields (a) to (d) and differential equations (1) to (4).

1. $\frac{dy}{dx} = \cos(x)$ has its direction field in ____**(d)**____.
2. $\frac{dy}{dx} = e^y$ has its direction field in ____**(c)**____.
3. $\frac{dy}{dx} = -y$ has its direction field in ____**(a)**____.
4. $\frac{dy}{dx} = x$ has its direction field in ____**(b)**____.

(b) (4 points) For each of the direction field (a) to (d), draw the solution passing through $(-2, -4)$.

(c) (4 points) For each of the direction field (a) to (d), draw the solution passing through $(-2, -2)$.

4. (10 points) Consider the initial value problem

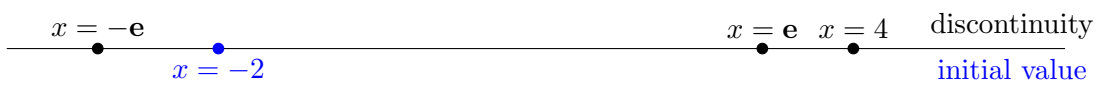
$$(x^2 - \mathbf{e}^2)y' + (x+2)y = \frac{\cos(2x)}{x+4} \quad \text{with} \quad y(-2) = 5.$$

Without solving the equation, what is the largest interval for $x$ in which a unique solution is guaranteed to exist? (Hint: $\mathbf{e} = 2.71828...$)

**Solution:** Here one needs to rewrite the DE as $y' + p(x)y = q(x)$ first;

$$y' + \frac{(x-2)}{(x^2-\mathbf{e}^2)}y = \frac{\cos(2x)}{(x-4)(x^2-\mathbf{e}^2)}.$$

Now $p(x) = \frac{(x-2)}{(x^2-\mathbf{e}^2)}$ is not continuous at $x = \mathbf{e}, -\mathbf{e}$ and $q(x) = \frac{\cos(2x)}{(x-4)(x^2-\mathbf{e}^2)}$ is not continuous at $x = 4, \mathbf{e}, -\mathbf{e}$. Therefore, the existence uniqueness theorem for linear DE guarantees that for $x \in (-\mathbf{e}, \mathbf{e})$ the unique solution exists.




5. (10 points) Using the separation of variables method, solve the differential equation

$$\frac{dy}{dx} = \frac{x}{y - x^2 y} \quad \text{with} \quad y(0) = \sqrt{2410}.$$

Also, write the largest possible interval for which the solution is defined.

**Solution:** We first separate the variables;

$$ydy = \frac{x}{1-x^2}dx$$

and then integrate both sides

$$\int ydy = \int \frac{x}{1-x^2}dx.$$

We then have

$$\frac{y^2}{2}(x) = -\frac{\ln|1-x^2|}{2} + C \quad \text{or} \quad y^2(x) = -\ln|1-x^2| + C$$

Since $y(0) = \sqrt{2410}$ we get $C = 2410$ and therefore

$$y^2(x) = -\ln|1-x^2| + 2410.$$

Note that $\ln|1-x^2|$ is defined when $1-x^2$ is positive. Therefore, $-1 < x < 1$ is the largest possible interval for which solution is defined.

6. (10 points) Solve the given initial value problem

$$xy' + y = e^x \quad \text{with} \quad y(1) = 2$$

and give the largest interval $I$ on which the solution is defined.

**Solution:** If we divide the whole equation by $x$ (assuming $x \neq 0$) we get

$$y' + \frac{1}{x}y = \frac{e^x}{x}.$$

Here we see that $\mu(x) = e^{\int \frac{1}{x}dx}$ is an integrating factor. We see that $\mu(x) = x$ is an integrating factor. Then multiplying the differential equation we get

$$xy' + y = e^x \quad \text{i.e.} \quad \frac{d}{dx}(xy) = e^x.$$

From this we see that

$$xy = e^x + c \quad \text{equivalently} \quad y(x) = \frac{e^x}{x} + \frac{c}{x}$$

is the general solution when $n \neq 0$. Since $y(1) = 2$ we get

$$2 = e + c \quad \text{i.e.,} \quad c = 2 - e$$

and the particular solution is

$$y(x) = \frac{e^x}{x} + \frac{2-e}{x}.$$

Notice the coefficients $\frac{1}{x}$ and $\frac{e^x}{x}$ are both continuous functions except $x = 0$. Therefore, the general solutions are defined on $(-\infty, 0)$ and $(0, \infty)$. Since 1 is in $(0, \infty)$ we see that the solution $y(x) = \frac{e^x}{x} + \frac{2-e}{x}$ for interval $I = (0, \infty)$ and it is the largest possible.

7. Consider the autonomous equation

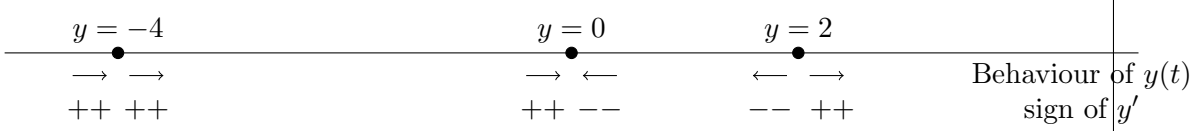
$$y' = y(y-2)(y+4)^2$$

(a) (4 points) Find all equilibrium solutions.

**Solution:** Solution: Here $f(y) = y(y-2)(y+4)^2$ and $f(y) = 0$ has solutions $y = 0, 2, -4$. Therefore, $y = 0, 2, -4$ are the all equilibrium solutions.

(b) (6 points) Classify the stability of each equilibrium solution as asymptotically stable(attractor), semi-stable, or unstable(repeller).

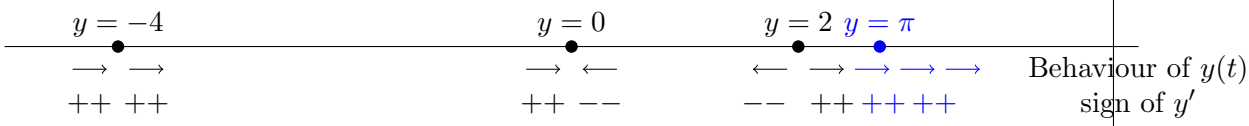
**Solution:** Solution:




Interpreting the figure; (close) solutions above $y = 2$ equilibrium solution runaway from $y = 2$, similarly solutions below $y = 3$ equilibrium solution runaway from $y = 2$. Therefore, $y = 2$ is unstable equilibrium solution. Solutions above $y = 0$ equilibrium solution approach to $y = 0$ and solutions below $y = 0$ equilibrium solution approach to $y = 0$ equilibrium solution. Therefore, $y = 0$ stable (or asymptotically stable) equilibrium solution. Solutions above $y = -4$ equilibrium solution runaway from $y = -4$ and solutions below $y = -3$ equilibrium solution approach to $y = -3$. Hence $y = -3$ is a semi-stable equilibrium solution.

(c) (3 points) If $y(22/7) = \pi$, what is $\lim_{t\to\infty} y(t)$? (Hint: $\pi = 3.14159265...$)

**Solution:**




From the figure, one should easily observe that as $y = \pi$, any solution with initial value $y(x_0) = \pi$ will approach to $\infty$ as $x \to \infty$. Therefore, $\lim_{x\to\infty} y(x) = \infty$.

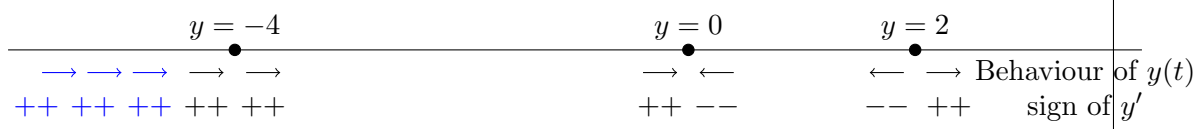
(d) (3 points) If $y(2\pi) = -3$, what is $y(t)$?

**Solution:** Since $y(x) = 0$ is an equilibrium solution then this solution stays on $y = 0$ forever, i.e, $y(t) = 0$. In particular, $y(3410) = 0$.

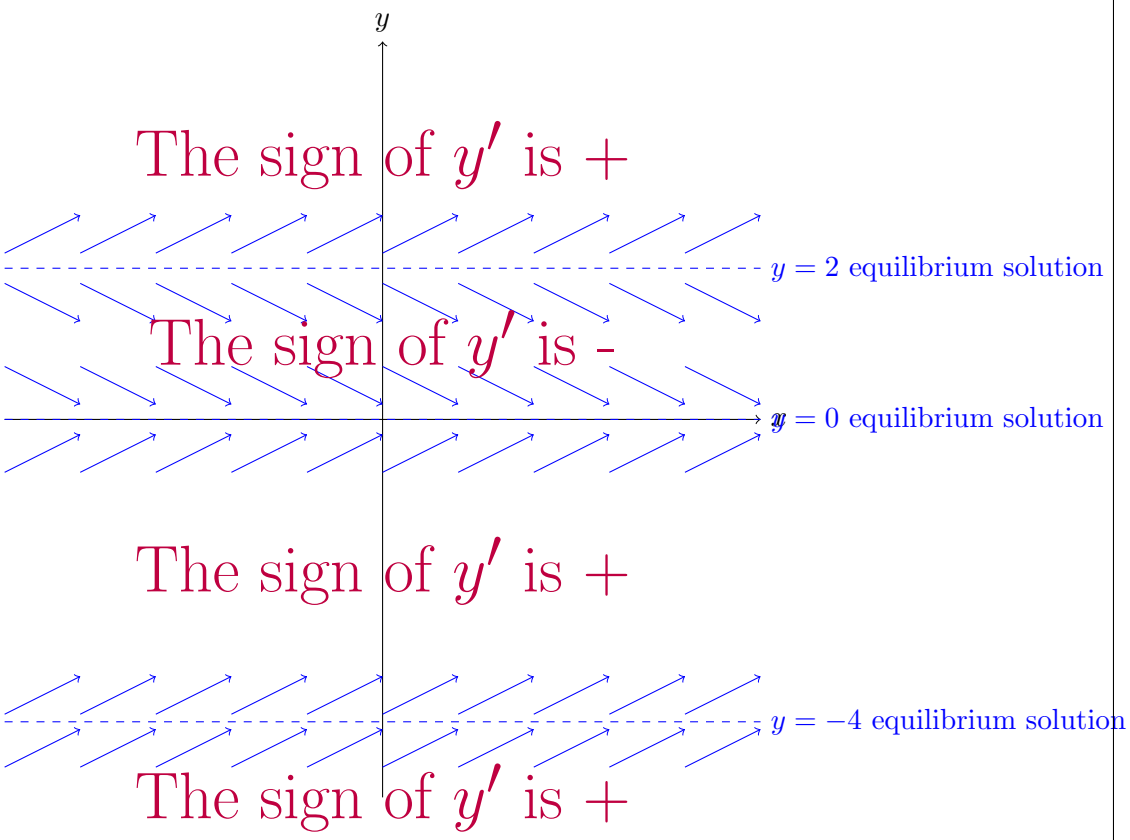
(e) (4 points) If $y(4) = \lambda$. Find intervals for $\lambda$ for which $\lim_{t\to\infty} y(t) = 0$.

**Solution:** Note that $y = -4$ is a semi-stable equilibrium solution therefore, solutions above $y = -4$ runs away from $y = -4$, the solutions below $y = -4$ will approach to

$y = -4$ equilibrium solution. Therefore, when $\lambda < -4$ and solution with $y(0) = \lambda$ will approach to the equilibrium solution $y = -4$. Hence any values of $\lambda \in (-\infty, -4)$. When $\lambda = -4$ then, $y = -4$ stays on that line, i.e, $y(x) = -4$ for all $x$. Hence the answer is $\lambda \in (-\infty, -4]$.




Indeed, the gradient field looks like;



8. (10 points) Solve the following Bernoulli equation

$$\frac{dy}{dx} + \frac{1}{3}y = e^x y^4.$$

with using the substitution $u = y^{-3}$.

**Solution:** Now the substitution we need to make is already given to us therefore, if we let

$$u = y^{-3} \quad \text{and} \quad \frac{du}{dx} = -3y^{-4}\frac{dy}{dx}$$

Divide the differential equation by $y^4$ to get

$$\frac{1}{y^4}\frac{dy}{dx} + \frac{1}{3y^3} = e^x$$

Now substituting $u$ and $u'$ we get

$$-\frac{1}{3}\frac{du}{dx} + \frac{u}{3} = e^x \quad \text{or} \quad \frac{du}{dx} - u = -3e^x$$

This differential equation is linear and we can find the integrating factor

$$\mu(x) = e^{-\int 1dx} = e^{-x}.$$

Multiplying the differential equation we get

$$e^{-x}\frac{du}{dx} - e^{-x}u = -3e^x e^{-x} = -3.$$

We know that

$$\frac{d}{dx}(e^{-x}u) = -3 \quad \text{equivalently} \quad e^{-x}u = -3x + c.$$

Now $u = 3xe^x + ce^x$ and

$$y(x) = u^{-1/3} = (3xe^x + ce^x)^{-1/3}$$

is the general solution.