Fall 2018 - Math 2410
Exam 2 - October 30
Time Limit: 75 Minutes

Name (Print): ____________________

This exam contains 8 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	16	
3	15	
4	23	
5	18	
6	18	
Total:	100	

1. (10 points) Consider the initial value problem

$$y' = 2x + y^2 \quad \text{with} \quad y(1) = 2.$$

Use Euler's method to obtain an approximation of $y(2)$ using the step size $h = .5$.

Solution: Here $x_0 = 1$ and since the step size is $h = .5$ we have $x_1 = 1.5$ and $x_2 = 2$. Also we let $f(x, y) = 2x - 3y + 1$. Then Euler's method gives us

$$y_{n+1} = y_n + hf(x_n, y_n).$$

where $y_{n+1} \approx y(x_{n+1})$.

$$y(x_1) \approx y(x_0) + f(x_0, y(x_0))h = 2 + .5(2 + 4) = 5.$$

Then

$$y(x_2) \approx y(x_1) + hf(x_1, y(x_1)) = 5 + .5(3 + 25) = 19$$

Hence $y(2) \approx 19$.

2. (16 points) Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between the temperature of the object itself and the temperature of its surroundings.

   A cup with water at 45 °C is placed in the cooler held at 5 °C. If after 2 minutes the water temperature is 25 °C, when will the water temperature be 15 °C?

Solution: We know that if $T(t)$ is the temperature of the cup then it satisfies

$$\frac{dT(t)}{dt} = -k(T(t) - T_s(t))$$

where $T_s(t) = 5$ and at $t = 0$ we have $T(0) = 45$. Therefore we have the following DE

$$\frac{dT(t)}{dt} + k(T(t) = 5k.$$

We then can find the integrating factor $\mu(t) = e^{\int k dt} = e^{kt}$ and if we multiply the DE we get

$$e^{kt}\frac{dT(t)}{dt} + e^{kt}k(T(t) = 5ke^{kt} \quad \text{or} \quad \frac{d}{dt}(e^{kt}T(t)) = 5ke^{kt}.$$

Integrating both sides with respect to t we get

$$e^{kt}T(t) = \int 5ke^{kt}dt = 5e^{kt} + c \quad \text{or} \quad T(t) = 5 + ce^{-kt}.$$

We know that $T(0) = 45$ we see that

$$45 = T(0) = 5 + c \quad \text{hence} \quad c = 40.$$

Therefore, $T(t) = 5 + 40e^{-kt}$.

On the other hand, we also know that after 2 minutes the water temperature is 25 °C, this tells us that $T(2) = 25$. Hence

$$25 = T(2) = 5 + 40e^{-2k} \quad \text{or} \quad e^{-2k} = \frac{1}{2}.$$

From this we get

$$-2k = \ln\frac{1}{2} = -\ln 2 \quad \text{or} \quad k = \frac{\ln 2}{2}.$$

Hence $T(t) = 5 + 40e^{-\frac{\ln 2}{2}t}$. We want to find the time at which temperature is 15,

$$15 = T(t) = 5 + 40e^{-\frac{\ln 2}{2}t} \quad \text{or} \quad \frac{1}{4} = e^{-\frac{\ln 2}{2}t},$$

which is

$$-\ln 4 = -\frac{\ln 2}{2}t \quad \text{or} \quad -2\ln 2^2 = -t\ln 2$$

which gives $t = 4$. Hence $T(4) = 15$ and it takes 4 minutes that the cup will have temperature 15 °C.

3. Given that $y_1(x) = x^2 - 2x$ and $y_2(x) = xe^{-x}$ are both solutions of a certain second order homogeneous equation
$$y'' + p(x)y' + q(x)y = 0$$
for some nice $p(x)$ and $q(x)$.

Answer each of the following questions. Show your work in each part.

(a) (6 points) Determine the Wronskian of y_1 and y_2, $W(y_1, y_2)$.

Solution: We know that

$$\begin{aligned} W(y_1(x), y_2(x)) = W(x^2 - 2x, xe^{-x}) &= \det \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} \\ &= \det \begin{pmatrix} x^2 - 2x & xe^{-x} \\ 2x - 2 & e^{-x} - xe^{-x} \end{pmatrix} \\ &= (x^2 - 2x)(e^{-x} - xe^{-x}) - (2x - 2)xe^{-x} \\ &= -x^2 e^{-x}(x - 1) \end{aligned}$$

(b) (3 points) Determine largest possible intervals for which y_1 and y_2 form a set of fundamental solutions of this equation.

Solution: As we observe from part (a) that $W(y_1(x), y_2(x)) = -x^2e^{-x}(x-1)$ which vanishes at $x = 0$ and $x = 1$. We also know that if $W(y_1(x), y_2(x)) \neq 0$ in an interval then y_1 and y_2 form a set of fundamental solution. Hence all possible intervals are intervals which do not contain 0 and 1,

$$(-\infty, 0) \cup (0, 1) \cup (1, \infty).$$

$W(y_1(x), y_2(x))$ does not vanish any of those three intervals and they are the largest possible intervals for which y_1 and y_2 form a set of fundamental solutions of this equation.

(c) (6 points) True or false: $y_4(x) = 2x^2 - 4x + 2xe^{2-x}$ is also a solution. You must show your explanation.

Solution: This is true. As

$$y_4(x) = 2x^2 - 4x + 2xe^{2-x} = 2(x^2 - 2x) + 2e^2 xe^{-x} = 2y_1(x) + 2e^2 y_2(x).$$

Since we have a linear second order homogeneous equation above then any linear combination of solutions will also be a solution. Therefore, yes, $y_4(x) = 2x^2 - 4x + 2xe^{2-x}$ is also a solution.

4. Consider the given non-homogeneous differential equation

$$y'' - 4y' + 4y = 2x^2 + 1 + e^x.$$

(a) (6 points) Find the complementary solution to the homogeneous DE

$$y'' - 4y' + 4y = 0.$$

Solution: Since this is a second order linear DE with constant coefficients we first find the corresponding characteristic equation

$$r^2 - 4r + 4 = 0 \quad \text{or} \quad (r-2)^2 = 0.$$

Hence we have a double root and in this case the complementary solutions is

$$y_c(x) = c_1 e^{2x} + c_2 x e^{2x}.$$

(b) (12 points) Using the method of undetermined coefficients find the particular solution to the non-homogeneous DE

$$y'' - 4y' + 4y = 2x^2 + 1 + e^x.$$

Solution: (This is not the unique way, you can consider the right hand side as a single function). We can use the super position principle for non-homogeneous equation. Suppose y_{p_1} solves

$$y'' - 4y' + 4y = 2x^2 + 1$$

and y_{p_2} solves the remaining of the right hand side

$$y'' - 4y' + 4y = e^x.$$

Then $y_p = y_{p_1} + y_{p_2}$. Let us focus on y_{p_1}. We know that the y_{p_1} has the form $y_{p_1}(x) = ax^2 + bx + c$ which solves

$$y'' - 4y' + 4y = 2x^2 + 1.$$

Hence,e we need to find $y'_{p_1}(x)$ and $y''_{p_1}(x)$;

$$y_{p_1}(x) = ax^2 + bx + c, \quad y'_{p_1}(x) = 2ax + b, \quad y''_{p_1}(x) = 2a.$$

Substituting these into the DE we get

$$2a - 4(2ax + b) + 4(ax^2 + bx + c) = 2x^2 + 1.$$

Combining x^2 terms, x terms, and constant terms on the left hand side we get

$$4ax^2 + (4b - 8a)x + 2a - 4b + 4c = 2x^2 + 1.$$

From this we get $4a = 2$, i.e., $a = 1/2$, $4b - 8a = 0$, i.e., $b = 1$. Finally, $2a - 4b + 4c = 1$ gives $1 - 4 + 4c = 1$, i.e., $c = 1$. We get $y_{p_1}(x) = \frac{1}{2}x^2 + x + 1$.
We now focus on $y_{p_2}(x)$. Since e^x does not appear on the complementary solution the particular solution will have the following form

$$y_{p_2}(x) = Ae^x$$

which solves

$$y'' - 4y' + 4y = e^x.$$

To find A, we see $y'_{p_2}(x) = Ae^x$ and $y''(x) = Ae^x$. Hence

$$Ae^x - 4Ae^x + 4Ae^x = e^x \quad \text{or} \quad Ae^x = e^x.$$

We get $A = 1$. Therefore,

$$y_{p_2}(x) = e^x.$$

Combining our work

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x) = \frac{1}{2}x^2 + x + 1 + e^x.$$

(c) (5 points) First give the general solution of the DE and then find the solution satisfying the boundary conditions

$$y(0) = 3 \quad \text{and} \quad y(1) = 2e^2 + e + \frac{5}{2}.$$

Solution: The general solution is

$$y(x) = y_c(x) + y_p(x) = c_1e^{2x} + c_2xe^{2x} + \frac{1}{2}x^2 + x + 1 + e^x.$$

Since

$$y(0) = 3 = c_1 + 0 + 0 + 1 + 1 \quad \text{hence } c_1 = 1.$$

Now

$$y(1) = 2e^2 + e + \frac{5}{2} = e^2 + c_2e^2 + \frac{5}{2} + e \quad \text{hence } c_2 = 1.$$

Therefore,

$$y(x) = e^{2x} + xe^{2x} + \frac{1}{2}x^2 + x + 1 + e^x.$$

5. (18 points) Find a differential equation that has

$$y(x) = c_1 e^{\frac{x}{2}} + c_2 x e^{\frac{x}{2}} + 12 + 4x^2 e^{\frac{x}{2}}$$

as its general solution.

Solution: We should first observe that

$$y(x) = y_c(x) + y_p(x) = c_1 e^{\frac{x}{2}} + c_2 x e^{\frac{x}{2}} + 12 + 4x^2 e^{\frac{x}{2}}.$$

We first observe that, in the complementary solution $y_c(x) = c_1 e^{\frac{x}{2}} + c_2 x e^{\frac{x}{2}}$, the function $e^{\frac{x}{2}}$ is repeated. Therefore, $r = 1/2$ is a repeated root. The characteristic equation must be

$$(r - \frac{1}{2})^2 = r^2 - r + \frac{1}{4}.$$

Then the corresponding differential equation is

$$y'' - y' + \frac{1}{4} y = 0.$$

We now focus on the particular solution. Since $y_p(x) = 12 + 4x^2 e^{\frac{x}{2}}$ then $y_p'(x) = 8xe^{\frac{x}{2}} + 2x^2 e^{\frac{x}{2}}$ and $y''(x) = 8e^{\frac{x}{2}} + 4xe^{\frac{x}{2}} + 4xe^{\frac{x}{2}} + x^2 e^{\frac{x}{2}} = 8e^{\frac{x}{2}} + 8xe^{\frac{x}{2}} + x^2 e^{\frac{x}{2}}$. Hence

$$\begin{aligned} y_p'' - y_p' + \frac{1}{4} y_p &= e^{\frac{x}{2}}(8 + 8x + x^2) - e^{\frac{x}{2}}(8x + 2x^2) - \frac{1}{4}(12 + 4x^2 e^{\frac{x}{2}}) \\ &= 3 + 8e^{\frac{x}{2}}. \end{aligned}$$

From this we get that

$$y'' - y' + \frac{1}{4} y = 3 + 8e^{\frac{x}{2}}$$

is a differential equation whose solution is given as above.

6. (18 points) It is given that $y_1(x) = x$ and $y_2(x) = x^3$ are linearly independent solutions of

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 0 \quad \text{for } x > 0.$$

Using variations of parameter, find the general solution of the following non-homogeneous DE

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 4x\ln x.$$

(Hint: $\int 2x \ln x dx = x^2 \ln x - \frac{x^2}{2} + C$)

Solution: Notice that the coefficients are not constant hence we have to use the method from 4.6. To this end, we need to find Wronskian $W(y_1(x), y_2(x))$ first.

$$\begin{aligned} W = W(y_1(x), y_2(x)) = W(x, x^3) &= \det\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} \\ &= \det\begin{pmatrix} x & x^3 \\ 1 & 3x^2 \end{pmatrix} \\ &= 3x^3 - x^3 = 2x^3. \end{aligned}$$

Now we will consider particular solutions which has the following forms

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 satisfies the differential equation $u_1' = W_1/W$ and u_2 satisfies the differential equation $u_2' = W_2/W$ with

$$W_1 = \det\begin{pmatrix} 0 & y_2(x) \\ f(x) & y_2'(x) \end{pmatrix} = \det\begin{pmatrix} 0 & x^3 \\ 4x\ln x & 3x^2 \end{pmatrix}, \quad W_2 = \det\begin{pmatrix} y_1(x) & 0 \\ y_1'(x) & f(x) \end{pmatrix} = \det\begin{pmatrix} x & 0 \\ 1 & 4x\ln x. \end{pmatrix}$$

Hence $W_1 = -4x^4 \ln x$ and $W_2 = 4x^2 \ln x$. Now

$$u_1' = \frac{W_1}{W} = \frac{-4x^4\ln x}{2x^3} = -2x\ln x \quad \text{and} \quad u_2' = \frac{W_2}{W} = \frac{4x^2\ln x}{2x^3} = 2\frac{\ln x}{x}.$$

Then

$$u_1(x) = -2\int x\ln x dx = -x^2\ln x + \frac{x^2}{2}$$

and

$$u_2(x) = (\ln x)^2.$$

Hence

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) = -x^3\ln x + \frac{x^3}{2} + x^3(\ln x)^2.$$

Hence the general solution is

$$y(x) = y_c(x) + y_p(x) = c_1y_1(x) + c_2y_2(x) + y_p(x) = c_1x + c_2x^3 - x^3\ln x + \frac{x^3}{2} + x^3(\ln x)^2.$$