UCONN
UNIVERSITY OF CONNECTICUT

Fall 2018 - Math 2410
Exam 3 - December 6
Time Limit: 75 Minutes

Name (Print): ____________________

This exam contains 10 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	30	
2	16	
3	18	
4	18	
5	18	
6	0	
7	0	
Total:	100	

1. (a) (8 points) Find $\mathcal{L}\{f(t)\}$ where

$$f(t) = \begin{cases} e^{-t} & \text{when } 0 \le t < 5, \\ -1 & \text{when } t \ge 5 \end{cases}$$

Solution: One way to find this is using the definition of Laplace transform we get

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)dt = \int_0^5 e^{-st} f(t)dt + \int_5^\infty e^{-st} f(t)dt \\ &= \int_0^5 e^{-st} e^{-t} dt + \int_5^\infty e^{-st}(-1)dt \\ &= \int_0^5 e^{-(s+1)t} dt - \int_5^\infty e^{-st} dt \\ &= -\frac{1}{s+1} e^{-(s+1)t}\Big|_{t=0}^{t=5} + \frac{1}{s} e^{-st}\Big|_{t=5}^{\infty} \\ &= -\frac{1}{s+1} e^{-5(s+1)} + \frac{1}{s+1} - \frac{1}{s} e^{-5s}. \end{aligned}$$

(b) (6 points) Find $\mathcal{L}\{3t^2 + e^2 \sin(2t)\}$.

Solution: Using linearity of the Laplace transformation we get

$$\begin{aligned} \mathcal{L}\{3t^2 + \sin(2t)\} &= 3\mathcal{L}\{t^2\} + e^2 \mathcal{L}\{\sin(2t)\} \\ &= 3\frac{2!}{s^3} + \frac{2e^2}{s^2+4}. \end{aligned}$$

(c) (8 points) Find the integral

$$\int_0^\infty e^{-(s+4)t}\cos(2t)dt \qquad \text{(assuming } s > 0\text{)}.$$

Solution: Now if we look carefully the integrand can be separated as

$$\begin{aligned}
\int_0^\infty e^{-(s+4)t}\cos(2t)dt &= \int_0^\infty e^{-st}e^{-4t}\cos(2t)dt \\
&= \mathcal{L}\{e^{-4t}\cos(2t)\} \\
&= \mathcal{L}\{\cos(2t)\}|_{s\to s+4} \\
&= \frac{s}{s^2+4}|_{s\to s+4} \\
&= \frac{s+4}{(s+4)^2+4}.
\end{aligned}$$

(d) (8 points) Find $\mathcal{L}\{(16-t^2)\mathcal{U}(t-4)\}$ where $\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a, \\ 1, & t \ge a. \end{cases}$

Solution:

$$\begin{aligned}
F(s) &= \mathcal{L}\{(16-t^2)\mathcal{U}(t-4)\} \\
&= \mathcal{L}\{(16-((t-4)+4)^2)\mathcal{U}(t-4)\} \\
&= e^{-4s}\mathcal{L}\{16-(t+4)^2\} \\
&= e^{-4s}\mathcal{L}\{16-16-8t-t^2\} \\
&= e^{-4s}\mathcal{L}\{-8t-t^2\} \\
&= -8\frac{e^{-4s}}{s^2} - \frac{2e^{-4s}}{s^3}.
\end{aligned}$$

2. (a) (8 points) Given

$$F(s) = \frac{3s-3}{s^2+2s+10}.$$

Find $\mathcal{L}^{-1}\{F(s)\}$.

Solution: Since

$$\frac{3s-3}{s^2+2s+10} = \frac{3s-3}{(s+1)^2+9} = 3\frac{s+1}{(s+1)^2+9} - \frac{6}{(s+1)^2+9}$$

we get

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\{3\frac{s+1}{(s+1)^2+9} - \frac{6}{(s+1)^2+9}\}\\ &= 3\mathcal{L}^{-1}\{\frac{s+1}{(s+1)^2+9}\} - 2\mathcal{L}^{-1}\{\frac{3}{(s+1)^2+9}\}\\ &= 3e^{-t}\cos(3t) - 2e^{-t}\sin(3t).\end{aligned}$$

(b) (8 points) Given

$$G(s) = \frac{4e^{-s}}{s^4+4s^2}.$$

Find $\mathcal{L}^{-1}\{G(s)\}$.

Solution:

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{4e^{-s}}{s^4+4s^2}\} &= \mathcal{L}^{-1}\{e^{-s}\frac{4}{s^2(s^2+4)}\}\\ &= \mathcal{L}^{-1}\{e^{-s}(\frac{1}{s^2} - \frac{1}{s^2+4})\} = \mathcal{L}^{-1}\{e^{-s}\frac{1}{s^2}\} - \mathcal{L}^{-1}\{e^{-s}\frac{1}{s^2+4}\}\\ &= t|_{t\to t-1}\mathcal{U}(t-1) + \frac{1}{2}\sin(2t)|_{t\to t-1}\mathcal{U}(t-1)\\ &= (t-1)\mathcal{U}(t-1) + \frac{1}{2}\sin(2t-2)\mathcal{U}(t-1)\end{aligned}$$

3. Suppose that $\mathcal{L}\{f(t)\} = F(s) = \frac{3s^2}{s^5+1}$ for some function $f(t)$ with given that $f(0) = 1$, $f'(0) = -6$, and $f''(0) = 1$.

   (a) (4 points) Find $\mathcal{L}\{tf(t)\}$.

   Solution: Since
   $$\mathcal{L}\{tf(t)\} = (-1)\frac{d}{ds}F(s) = -\frac{6s^6 + 6s - 15s^6}{(s^5+1)^2} = \frac{9s^6 - 6s}{(s^5+1)^2}.$$

   (b) (3 points) Find $\mathcal{L}\{f''(t))\}$.

   Solution: Since
   $$\mathcal{L}\{f''(t))\} = s^2F(s) - sf(0) - f'(0) = \frac{3s^4}{s^5+1} - s + 6$$

(c) (5 points) Let $\mathcal{L}\{g(t)\} = \frac{s^5+1}{s^3}$. Find $f * g$.
(Hint: you may not need to use the integral definition of convolution).

Solution: Since

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = \frac{3s^2}{s^5+1}\frac{s^5+1}{s^3} = \frac{3}{s}.$$

Hence

$$f * g = \mathcal{L}^{-1}\{\frac{3}{s}\} = 3.$$

(d) (6 points) (Recall that $\mathcal{L}\{f(t)\} = F(s) = \frac{3s^2}{s^5+1}$ for some function $f(t)$ with given that $f(0) = 1$, $f'(0) = -6$, and $f''(0) = 1$).
Let $g(t) = \mathcal{L}^{-1}\{e^{-6s}\frac{3s^2}{s^5+1}\}$. Find $g(6)$.

Solution: Since

$$f(t) = \mathcal{L}^{-1}\{\frac{3s^2}{s^5+1}\}$$

we get

$$g(t) = \mathcal{L}^{-1}\{e^{-6s}\frac{3s^2}{s^5+1}\} = \mathcal{U}(t-6)f(t-6).$$

Hence

$$g(6) = \mathcal{U}(0)f(0) = 0 \cdot 1 = 0.$$

4. (18 points) Using Laplace transform method solve the following DE

$$y'(t) + 3\int_0^t y(t-\tau)\cos(\tau)d\tau = 0 \quad \text{with } y(0) = 1.$$

Solution: If we take the Laplace transform of both sides and use linearity we get

$$\mathcal{L}\{y'(t) + 3\int_0^t y(t-\tau)\cos(\tau)d\tau\} = \mathcal{L}\{0\}$$

from which we get

$$\mathcal{L}\{y'(t)\} + 3\mathcal{L}\{\int_0^t y(t-\tau)\cos(\tau)d\tau\} = 0$$

Let $Y(s) = \mathcal{L}\{y(t)\}$. Then using the derivative formula we get

$$sY(s) - y(0) + 3\mathcal{L}\{\int_0^t y(t-\tau)\cos(\tau)d\tau\} = 0.$$

Now notice that

$$\int_0^t y(t-\tau)\cos(\tau)d\tau = y(t) * \cos(t)$$

and therefore

$$\mathcal{L}\{\int_0^t y(t-\tau)\cos(\tau)d\tau\} = \mathcal{L}\{y(t) * \cos(t)\} = \mathcal{L}\{y(t)\}\mathcal{L}\{\cos(t)\} = \frac{sY(s)}{s^2+1}.$$

Hence we get

$$sY(s) - 1 + 3\frac{sY(s)}{s^2+1} = 0.$$

Now if we solve for $Y(s)$ we get (and we do partial fractional decomposition)

$$Y(s) = \frac{s^2+1}{s^3+4s} = \frac{A}{s} + \frac{Bs+C}{s^2+4}.$$

If we solve for A, B, and C we get $A = 1/4$, $B = 3/4$, and $C = 0$. Hence

$$Y(s) = \frac{1}{4s} + \frac{3s}{4(s^2+4)}.$$

Now we take the inverse Laplace transform to get

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\frac{1}{4s} + \frac{3s}{4(s^2+4)}\} \\ &= \frac{1}{4} + \frac{3}{4}\cos(2t). \end{aligned}$$

5. (18 points) Using Laplace transform method solve the following DE

$$y'' + 4y' + 4y = 2te^{-2t} \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = -2.$$

Solution: Take the Laplace transform of both sides to get

$$\mathcal{L}\{y'' + 4y' + 4y\} = \mathcal{L}\{2te^{-2t}\}$$

Using linearity and derivative formulas we get

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 2\mathcal{L}\{te^{-2t}\} = \frac{2}{(s+2)^2}.$$

We let $Y(s) = \mathcal{L}\{y(t)\}$ and use derivative formulas we have

$$s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 4Y(s) = \frac{2}{(s+2)^2}.$$

We then solve for $Y(s)$ to get

$$Y(s) = \frac{s+2}{(s+2)^2} + \frac{2}{(s+2)^4}.$$

Then

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\} &= \mathcal{L}^{-1}\{\frac{1}{(s+2)} + \frac{2}{(s+2)^4}\} \\ &= e^{-2t} + \frac{2}{6}t^3e^{-2t}. \end{aligned}$$

6. **LRC-Series Circuits**. The charge on the capacitor is related to the current $i(t)$ by $i = dq/dt$ which satisfies

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t).$$

Let $L = 5\,\mathrm{h}$, $R = 20\Omega$, $C = \frac{1}{20}\,\mathrm{f}$, $E(t) = 4\,\mathrm{V}$, $q(0) = 0\,\mathrm{C}$, $i(0) = 5\,\mathrm{A}$.

(a) (5 points (bonus)) Find the charge $q(t)$.

Solution: Now the equation becomes

$$5q'' + 20q' + 20q = 4.$$

Hence we need to solve the DE. We can use Laplace transform or other methods. We can find first the characteristic equation and then the roots

$$5r^2 + 20r + 20 = 0 \quad \text{or} \quad r = -2 \text{ double root.}$$

Hence the complementary solution is

$$q(t) = c_1e^{-2t} + c_2te^{-2t}.$$

Particular solution is $q_p(t) = 4/20 = 1/5$. Hence

$$q(t) = c_1e^{-2t} + c_2te^{-2t} + \frac{1}{5}.$$

Since $q(0) = 0$ we get $c_1 = -1/5$ and $q'(0) = i(0) = 5$ we get

$$q'(0) = \frac{2}{5} + c_2 = 5 \quad \text{i.e.} \quad c_2 = 3.$$

Hence

$$q(t) = -\frac{1}{5}e^{-2t} + \frac{23}{5}te^{-2t} + \frac{1}{5}.$$

7. (18 points (bonus)) Solve the following IVP

$$y'' + 2ty' - 4y = 1 \quad \text{with} \quad y(0) = 0 = y'(0).$$

Solution: Let $Y(s) = \mathcal{L}\{y(t)\}$. Take the Laplace transform

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{ty'\} - 4\mathcal{L}\{y\} = \mathcal{L}\{1\} = \frac{1}{s}.$$

and get

$$s^2Y(s) - sy(0) - y'(0) + 2(-1)\frac{d}{ds}(sY(s) - y(0)) - 4Y(s) = \frac{1}{s}.$$

We then get a first order

$$s^2Y(s) - 2Y(s) - 2sY'(s) - 4Y(s) = \frac{1}{s}.$$

After combining terms we get

$$-2sY'(s) + (s^2 - 6)Y(s) = \frac{1}{s} \quad \text{or} \quad Y'(s) + \frac{6 - s^2}{2s} = \frac{-1}{2s^2}.$$

From this we find integrating factor μ where

$$\mu(t) = e^{\int(-s + \frac{6}{2s})ds} = e^{-\frac{s^2}{4} + 3\ln s} = s^3e^{-\frac{s^2}{4}}.$$

Hence

$$\frac{d}{ds}[s^3e^{-\frac{s^2}{4}}Y(s)] = -s^3e^{-\frac{s^2}{4}}\frac{1}{2s^2} = -se^{-\frac{s^2}{4}}$$

Therefore,

$$s^3e^{-\frac{s^2}{4}}Y(s) = -\int se^{-\frac{s^2}{4}}ds = e^{-\frac{s^2}{4}} + c.$$

Hence

$$Y(s) = \frac{e^{-\frac{s^2}{4}}}{s^3e^{-\frac{s^2}{4}}} + \frac{c}{s^3e^{-\frac{s^2}{4}}} = \frac{1}{s^3} + \frac{c}{s^3e^{-\frac{s^2}{4}}}.$$

Remember that $Y(s) \to 0$ as $s \to \infty$. Then we should have $c = 0$. Hence

$$Y(s) = \frac{1}{s^3}.$$

Then

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{t^2}{2}.$$