UCONN

UNIVERSITY OF CONNECTICUT

Fall 2018 - Math 2410
Practice Exam 1 - September 18
Time Limit: 75 Minutes

Name (Print): ____________ Solution KEY

This exam contains 12 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
7	20	
8	10	
Total:	100	

1. For each part in (a) and (b), write down the letter corresponding to the equation on the list with the specified properties. Also answer parts (c), (d), and (e).

A. $y^{(2410)} + 5y + y' = \cos(y)$.
B. $\frac{dy}{dx} = \frac{6y}{x}$.
C. $\frac{dy}{dx} = 2y^3 - 16$.
D. $(y')^{2018} + 2y^{2410} = 0$.

(a) (4 points) First order linear differential equation which is separable equation.

Solution: First order and linear differential equation is B.

(b) (4 points) First order autonomous differential equation.

Solution: First order and autonomous is C. and D.

(c) (4 points) What is a suitable integrating factor that could be used to solve the linear differential equation you found in part (a)?

Solution: Since B. is the linear equation then if we rewrite this

$$\frac{dy}{dx} - \frac{6y}{x} = 0.$$

Then the integrating factor is

$$\mu(x) = e^{-6\int \frac{1}{x}dx} = e^{\ln x^{-6}} = \frac{1}{x^6}.$$

(d) (4 points) Write down the order of the differential equations in (A) to (D).
A. has order **2410**.
B. has order **1**.
C. has order **1**.
D. has order **1**.

(e) (4 points) Test the DE from (A) to (D) if they are linear or nonlinear.
A. is a **nonlinear**.
B. is a **linear**.
C. is a **nonlinear**.
D. is a **nonlinear**.

2. Consider the following differential equation

$$xy^2 + x^2 + (x^2y + y)y' = 0.$$

(a) (3 points) Is the differential equation exact?

Solution: Yes. To see this, we rewrite the differential equation

$$(xy^2 + x^2)dx + (x^2y + y)dy = 0/$$

Hence $M(x, y) = xy^2 + x^2$ and $N(x, y) = x^2y + y$ and they are both differentable functions everywhere we have

$$\frac{\partial M}{\partial y} = 2xy \qquad \text{and} \qquad \frac{\partial N}{\partial x} = 2xy.$$

Since $M_y = N_x$ then above DE is exact.

(b) (4 points) Find the 1-parameter family of solution of the differential equation (leave the solution as an implicit function).

Solution: We can rewrite the DE as

$$(xy^2 + x^2)dx + (x^2y + y)dy = 0.$$

From this, we want to find a function $f(x, y) = 0$ for which $f_y = x^2y + y$ and $f_x = xy^2 + x^2$. Hence

$$f(x, y) = \int f_y dy = \int (x^2y + y)dy = \frac{1}{2}x^2y^2 + \frac{1}{2}y^2 + g(x).$$

To find $g(x)$ we use the second information $f_x = xy^2 + x^2$;

$$f_x = xy^2 + x^2 = 2xy^2 + g'(x).$$

Hence $g'(x) = x^2$ and $g(x) = x^3/3 + c_1$. Substitute this in f to get

$$f(x, y) = \frac{1}{2}x^2y^2 + \frac{1}{2}y^2 + \frac{x^3}{3} + c_1 = 0$$

and this is an 1-parameter family of solution of the differential equation.

(c) (3 points) Find the particular solution to the initial value problem $y(0) = 2$.

Solution: As the initial value is $y(0) = 2$ means when $x = 0$ then $y = 2$. Therefore,

$$f(x, y) = f(0, 2) = 0 + 2 + 0 + c_1 = 0$$

which gives us $c_1 = -2$, and

$$f(x, y) = \frac{1}{2}x^2y^2 + \frac{1}{2}y^2 + \frac{x^3}{3} - 2 = 0$$

is the solution to the initial value problem. We can solve for y.

$$y^2(\frac{1}{2}x^2 + \frac{1}{2}) = 2 - \frac{x^3}{3} \quad \text{equaivalently} \quad y^2 = \frac{2 - \frac{x^3}{3}}{\frac{1}{2}x^2 + \frac{1}{2}}.$$

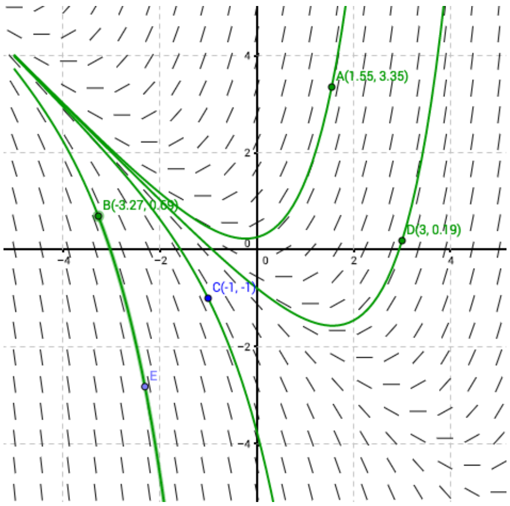
Now

$$y(x) = \pm\sqrt{\frac{2 - \frac{x^3}{3}}{\frac{1}{2}x^2 + \frac{1}{2}}}.$$

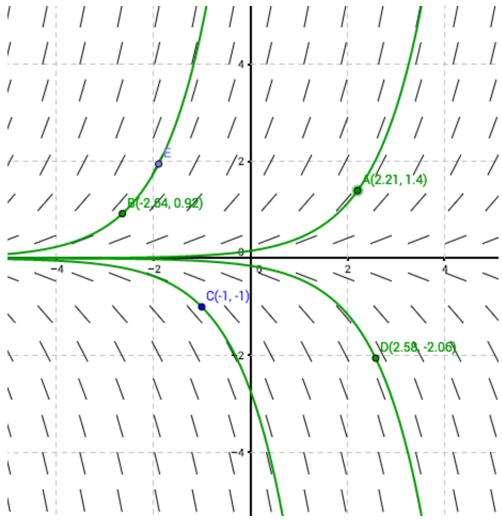
Since our initial value $y(0) = 2$ we then pick the one with $y > 0$;

$$y(x) = \sqrt{\frac{2 - \frac{x^3}{3}}{\frac{1}{2}x^2 + \frac{1}{2}}} = \sqrt{\frac{12 - 2x^3}{3(x^2 + 1)}}.$$

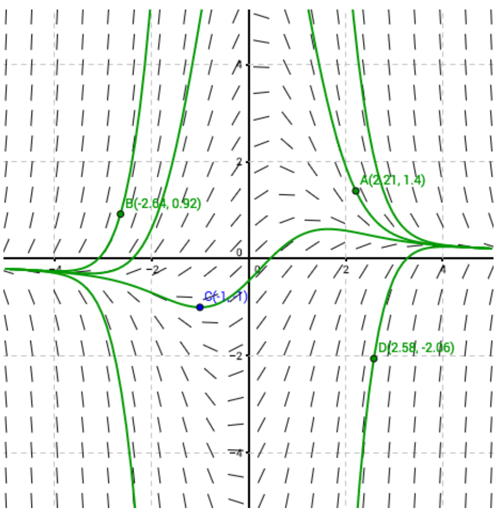
3. Consider the following direction fields (slope fields);



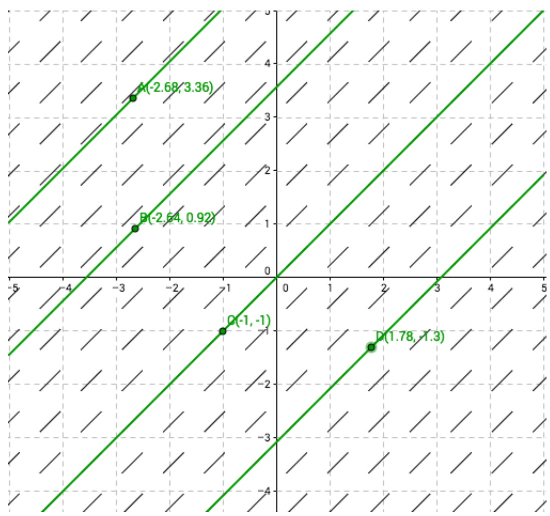
(a)



(b)



(c)



(d)

and the differential equations;

1. $\frac{dy}{dx} = y$ has its direction field in ______**(b)**______.
2. $\frac{dy}{dx} = 1 - xy$ has its direction field in ______**(c)**______.
3. $\frac{dy}{dx} = 1$ has its direction field in ______**(d)**______.
4. $\frac{dy}{dx} = x + y$ has its direction field in ______**(a)**______.

(a) (4 points) Match the given direction fields (a) to (d) and differential equations (1) to (4).

(b) (3 points) For each of the direction field (a) to (d), draw at least 3 solutions curves on the given graph.

(c) (3 points) For each of the direction field (a) to (d), draw the solution passing through $(-1, -1)$.

4. (10 points) Consider the initial value problem

$$(t^2-4)y' + \frac{t+2}{t}y = \frac{t^3}{t-5}, \quad y(4) = \frac{1}{2}.$$

Without solving the equation, what is the largest interval for t in which a unique solution is guaranteed to exist?

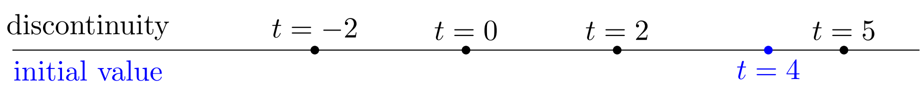
Solution: Here we are looking for an interval which contains 4. First, as we have t and $t-5$ in the denominator, $t \neq 5, 0$. Moreover, coefficient of y' should not be zero, from which we get $t \neq +2, -2$. Now, I can rewrite the DE (by dividing t^2-4)

$$y' + \frac{t+2}{(t^2-4)t}y = \frac{t^3}{(t-5)(t^2-4)}$$

Hence this is a linear first order DE with

$$p(t) = \frac{t+2}{(t^2-4)t} \qquad \text{and} \qquad q(t) = \frac{t^3}{(t-5)(t^2-4)}.$$

Now $p(t)$ is continuous except at $t = 0, 2$ and $q(t)$ is continuous except at $5, 2, -2$.



Therefore, the largest interval containing 4 and does not contain any discontinuity point is $(2, 5)$. By the existence uniqueness theorem for linear first order DE, we know that there exists a unique solution to above DE with any initial value for $t \in (2, 5)$.

5. (10 points) Using the separation of variables method, solve the differential equation

$$y' + 2x(y+1) = 0 \text{ with } y(0) = 2.$$

Solution: We can rewrite the DE as

$$\frac{dy}{dx} = -2x(y+1) \quad \text{or} \quad \frac{dy}{y+1} = -2xdx.$$

From this we integrate both sides to get

$$\ln|y+1| = -x^2 + c \quad \text{equivalently} \quad y+1 = e^{-x^2+c}.$$

If we solve for y we get the general solution

$$y(x) = -1 + e^{-x^2+c}.$$

To find the particular solution we use the given initial condition $y(0) = 2$. Therefore,

$$2 = y(0) = -1 + e^c \quad \text{hence} \quad e^c = 3.$$

Then the particular solution is

$$y(x) = -1 + e^{-x^2} e^c = -1 + 3e^{-x^2}.$$

6. Consider the following differential equation

$$2y' + y = e^x.$$

(a) (5 points) Find the 1-parameter family of solution of the differential equation. Write your solution in *explicit* form. (i.e., solve for y).

Solution: Since this is a first order linear differential equation, the first is to rewrite the DE in the form of $y' + P(x)y = Q(x)$;

$$y' + \frac{1}{2}y = \frac{1}{2}e^x.$$

Then we find the integrating factor $\mu(x)$

$$\mu(x) = e^{\int \frac{1}{2}dx} = e^{\frac{x}{2}}.$$

We then multiply the DE with the integrating factor to get

$$e^{\frac{x}{2}}y' + e^{\frac{x}{2}}\frac{1}{2}y = \frac{1}{2}e^x e^{\frac{x}{2}} = \frac{1}{2}e^{\frac{3x}{2}}.$$

Note that (the reason we multiplied the DE with integrating factor) we have

$$\frac{d}{dx}(e^{\frac{x}{2}}y) = \frac{1}{2}e^{\frac{3x}{2}}$$

From this we get

$$e^{\frac{x}{2}}y = \int \frac{1}{2}e^{\frac{3x}{2}}\,dx = \frac{2}{3}\frac{1}{2}e^{\frac{3x}{2}} + c = \frac{1}{3}e^{\frac{3x}{2}} + c$$

If we solve for y we get the general solution

$$y(x) = \frac{1}{3}e^x + ce^{\frac{-x}{2}}$$

(b) (2 points) Using part (a), find the solution of the differential equation with the given initial value $y(0) = \alpha$.

Solution: Since $y(x) = \frac{1}{3}e^x + ce^{\frac{-x}{2}}$ from this we get

$$\alpha = y(0) = \frac{1}{3} + c \quad \text{which gives} \quad c = \alpha - \frac{1}{3}.$$

Hence the particular solution is

$$y(x) = \frac{1}{3}e^x + (\alpha - \frac{1}{3})e^{\frac{-x}{2}}$$

(c) (3 points) For what value(s) of α, the solution you found in (b) remains finite as $x \to -\infty$?

Solution: Notice that as $x \to -\infty$ we have $e^x \to 0$ and $e^{-x/2} \to \infty$. If we want our solution to be finite then we need to pick $\alpha = 1/3$ so that

$$y(x) = \frac{1}{3}e^x$$

remains finite as $x \to -\infty$.

7. Consider the autonomous equation

$$y' = y^2(3-y)(3+y).$$

(a) (4 points) Find all equilibrium solutions.

Solution: Here $f(y) = y^2(3-y)(3+y) = 0$ has solutions $y = 0$, $y = 3$, $y = -3$ are the all equilibrium solutions.

(b) (6 points) Classify the stability of each equilibrium solution as asymptotically stable(attractor), semi-stable, or unstable(repeller).

Solution:

$y=-3$	$y=0$	$y=3$	
$\longleftarrow \quad \longrightarrow$	$\longrightarrow \quad \longrightarrow$	$\longrightarrow \quad \longleftarrow$	Behaviour of $y(t)$
$-- \quad ++$	$++ \quad ++$	$++ \quad --$	sign of y'

Therefore, (close) solutions above $y = 3$ equilibrium solution approaches to $y = 3$, similarly solutions below $y = 3$ equilibrium solution approach to $y = 3$. Therefore, $y = 3$ is stable (or asymptotically stable) equilibrium solution. Solutions above $y = 0$ equilibrium solution run away from $y = 0$ and solutions below $y = 0$ equilibrium solution approach to $y = 0$ equilibrium solution. Therefore, $y = 0$ semi-stable equilibrium solution. Solutions above $y = -3$ equilibrium solution runaway from $y = -3$ and similarly solutions below $y = -3$ equilibrium solution also runaway from $y = -3$. Hence $y = -3$ is an unstable equilibrium solution.

(c) (3 points) If $y(22/7) = \pi$, what is $\lim_{t\to\infty} y(t)$? (Hint: $\pi = 3.14159265...$)

Solution: Since $y(22/7) = \pi$ and $y_0 = \pi > 3$ and $y = 3$ is a stable solution

$$\lim_{t\to\infty} y(t) = 3.$$

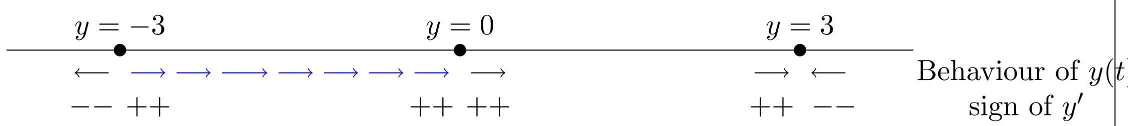
$y=-3$	$y=0$	$y=3$ $y=\pi$	
$\longleftarrow \quad \longrightarrow$	$\longrightarrow \quad \longrightarrow$	$\longrightarrow \quad \longleftarrow \quad \longleftarrow$	Behaviour of $y(t)$
$-- \quad ++$	$++ \quad ++$	$++ \quad ----$	sign of y'

(d) (3 points) If $y(2\pi) = -3$, what is $y(t)$?

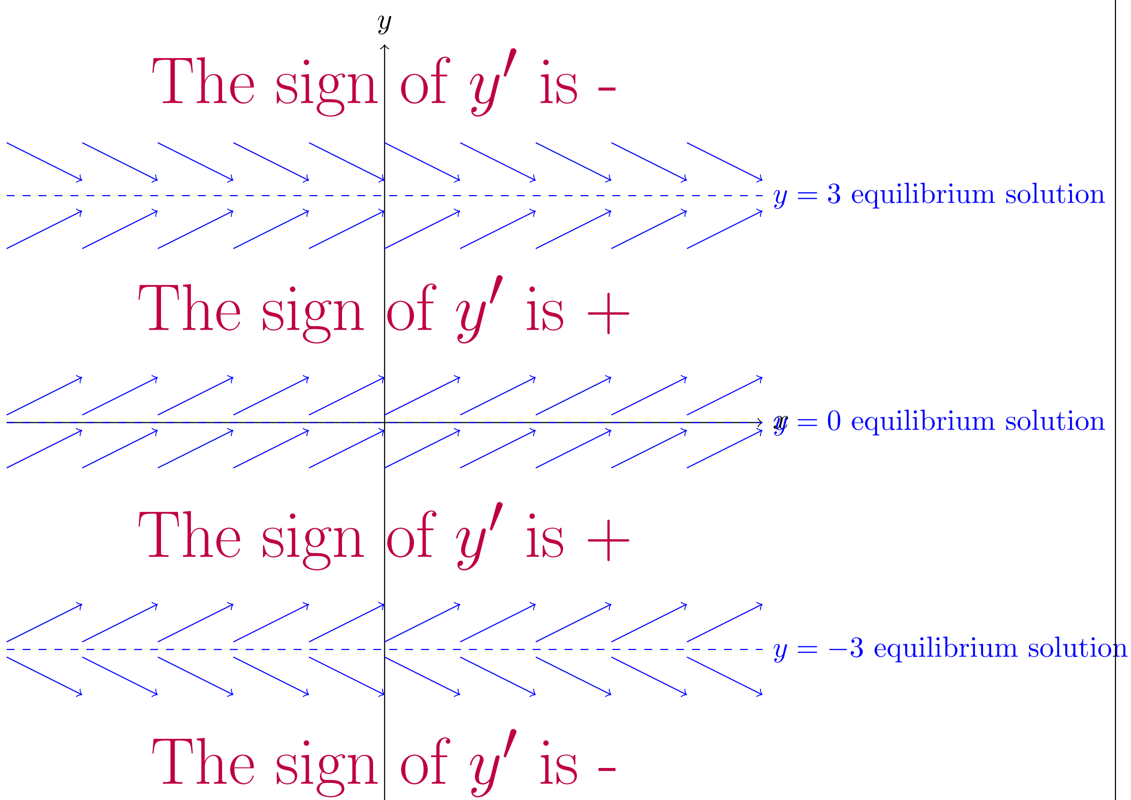
Solution: Since $y(t) = -3$ is an equilibrium solution then for every t, $y(t) = -3$.

(e) (4 points) If $y(4) = \lambda$. Find intervals for λ for which $\lim_{t\to\infty} y(t) = 0$.

Solution: Note that $y = 0$ is a semi-stable equilibrium solution therefore, for positive values of y_0 but close to zero, the solutions will approach to $y = 3$ stable equilibrium solutions. But for $-3 < y_0 < 0$, then all solutions will approach to $y = 0$ is an equilibrium solution. Therefore, $\lambda \in (-3, 0]$. (Notice that $y = 0$ is solution and $y = 0$ has limit 0 as $t \to \infty$.)



Indeed, one can verify above results by checking the gradient vector field for above autonomous differential equation;



8. (10 points) Solve the following Bernoulli equation

$$xy' + y + x^2y^2e^x = 0.$$

with using the substitution $u = y^{-1}$.

Solution: Since the substitution is already given to us, we just need to rewrite the DE in terms of u and u'. Since $u = y^{-1}$ then

$$u' = -y^{-2}y'.$$

We next rewrite the DE (dividing both sides by xy^2 assuming $x \neq 0$ and $y \neq 0$)

$$\frac{y'}{y^2} + \frac{1}{xy} + xe^x = 0$$

Using this and substituting y' we get

$$-u' + \frac{1}{x}u + xe^x = 0.$$

Now this is a linear first order DE. If we rewrite the DE we get $u' - \frac{1}{x}u = xe^x$. Then we find an integrating factor

$$\mu(x) = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = x^{-1}.$$

By multiplying the DE with $\mu(x) = x^{-1}$ we get

$$x^{-1}u' - \frac{1}{x^2} = e^x \quad \text{or} \quad \frac{d}{dx}(x^{-1}u) = e^x.$$

From this we get

$$x^{-1}u = e^x + c \quad \text{or} \quad u(x) = xe^x + xc.$$

Since $u = y^{-1}$ we get the general solution

$$y(x) = u^{-1}(x) = \frac{1}{xe^x + xc}.$$