

Fall 2018 - Math 2410 Practice Exam 2 - October 30 Time Limit: 75 Minutes Name (Print):

This exam contains 10 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- If you use a "fundamental theorem" you must indicate this and explain why the theorem may be applied.
- Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- Mysterious or unsupported answers will not receive full credit. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	12	
2	16	
3	15	
4	12	
5	15	
6	15	
7	15	
Total:	100	

1. (12 points) Consider the initial value problem

$$y' = 2x - 3y + 1$$
 with  $y(1) = 5$ .

Use Euler's method to obtain an approximation of y(1.2) using the step size h = .1.

**Solution:** Here  $x_0 = 1$  and since the step size is h = .1 we have  $x_1 = 1.1$  and  $x_2 = 1.2$ . Also we let f(x, y) = 2x - 3y + 1. Then Euler's method gives us

$$y_{n+1} = y_n + hf(x_n, y_n).$$

where  $y_{n+1} \approx y(x_n)$ .

$$y(x_1) \approx y(x_0) + f(x_0, y(x_0))h = 5 + .1f(1, 5) = 5 + .1(2 - 15 + 1) = 5 - 1.2 = 3.8.$$

Then

 $y(x_2) \approx y(x_1) + hf(x_1, y(x_1)) = 3.8 + .1f(1.1, 3.8) = 3.8 + .1(2.2 - 11.4 + 1) = 3.8 - .82 = 2.98.$ 

Hence  $y(1.2) \approx 2.98$ .

- 2. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between the temperature of the object itself and the temperature of its surroundings.
  - (a) (5 points) There is a cup of ice water in a room with ambient temperature  $T_s$ , which satisfies at time t,  $T_s(t) = 70 + e^{-t}\sin(t)$ . The initial temperature of the ice water is  $30^{\circ}F$ . Assume the absolute value of the proportionality constant k is 1. Let T(t) be the temperature of the ice water after time t. Write a differential equation with an initial value for T.

Solution: We know that

$$\frac{dT(t)}{dt} = -k(T(t) - T_s(t)) = -(T(t) - 70 - e^{-t}\sin(t))$$

and at t = 0 we have T(0) = 30.

(b) (6 points) What is the temperature of the ice water at time t?

**Solution:** We need to solve  $T' + T = 70 + e^{-t} \sin(t)$  with T(0) = 30. We can find the integrating factor  $\mu(t) = e^{\int dt} = e^t$ . Hence

$$\frac{d}{dt}(e^tT) = 70e^t + \underbrace{e^t e^{-t}\sin(t)}_{=\sin(t)}.$$

Hence  $e^t T(t) = 70e^t - \cos(t) + c$  or  $T(t) = 70 - e^{-t}\cos(t) + ce^{-t}$ . Since T(0) = 30 we have

$$T(0) = 30 = 70 - 1 + c$$
 hence  $c = -39$ .

Hence

$$T(t) = 70 - e^{-t}\cos(t) - 39e^{-t}$$

(c) (5 points) Determine  $\lim_{t\to\infty} T(t)$ . Show your work

Solution: Since  $T(t) = 70 - e^{-t}\cos(t) - 39e^{-t}$  we see that  $\lim_{t \to \infty} 70 - e^{-t}\cos(t) - 39e^{-t} = 70$ as  $e^{-t}$  to 0 as  $t \to \infty$ . 3. (a) (4 points) Verify that  $y_1(x) = e^{2x}$  and  $y_2(x) = e^{5x}$  are the solutions to y'' - 7y' + 10y = 0on  $(-\infty, \infty)$ .

**Solution:** For  $y_1(x) = e^{2x}$  we have  $y'_1(x) = 2e^2x$  and  $y''_1(x) = 4e^2x$ . Substituting this into the DE we get

$$4e^{2}x - 7(2e^{2}x) + 10e^{2x} = 14e^{2x} - 14e^{2x} = 0.$$

Hence  $y_1(x)$  is a solution. Similarly,  $y_2(x) = e^{5x}$  we have  $y'_2(x) = 5e^2x$  and  $y''_2(x) = 25e^2x$ . Substituting this into the DE we get

$$25e^{2}x - 7(5e^{2}x) + 10e^{2x} = 35e^{2x} - 35e^{2x} = 0.$$

Hence  $y_2(x)$  is also a solution.

(b) (4 points) Verify that  $y_1(x) = e^{2x}$  and  $y_2(x) = e^{5x}$  are linearly independent solutions of the above DE on  $(-\infty, \infty)$ .

**Solution:** To check that  $y_1(x) = e^{2x}$  and  $y_2(x) = e^{5x}$  are linearly independent, we need to show that the Wronskian of  $y_1$  and  $y_2$   $W(y_1(x), y_2(x)) = W(e^{2x}, e^{5x}) \neq 0$  for every  $x \in (-\infty, \infty)$ . Since

$$W(y_1(x), y_2(x)) = W(e^{2x}, e^{5x}) = \det \begin{pmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{pmatrix} = \det \begin{pmatrix} e^{2x} & e^{5x} \\ 2e^{2x} & 4e^{5x} \end{pmatrix}$$
$$= 4e^{5x}e^{5x} - 2e^{2x}e^{5x} = 2e^{7x} \neq 0$$

for every  $x \in (-\infty, \infty)$ . Hence we see that  $W(y_1(x), y_2(x)) = W(e^{2x}, e^{5x}) \neq 0$  for any  $x \in (-\infty, \infty)$ . This shows that  $y_1(x) = e^{2x}$  and  $y_2(x) = e^{5x}$  are linearly independent solutions  $(-\infty, \infty)$ .

(c) (4 points) Verify that  $y_p(x) = 6e^x$  is a particular solution to the DE

$$y'' - 7y' + 10y = 24e^x$$
 on  $(-\infty, \infty)$ .

- Solution: Since  $y_p(x) = 6e^x$  then  $y'_p(x) = 6e^x = y''_p(x) = 6e^x$ . Then  $y''_p - 7y'_p + 10y_p = 6e^x - 42e^x + 60e^x = 24e^x$ which shows that  $y_p(x) = 6e^x$  is a particular solution to the  $y'' - 7y' + 10y = 24e^x$ .
- (d) (3 points) Combining your work (a)-(c), conclude that  $y(x) = c_1 e^{2x} + c_2 e^{5x} + 6e^x$  is the general solution of the non-homogeneous DE

$$y'' - 7y' + 10y = 24e^x$$
 on  $(-\infty, \infty)$ .

Solution: From the theorem we saw in the class the general solution is

$$y(x) = y_c(x) + y_p(x) = c_1 e^{2x} + c_2 e^{5x} + 6e^x$$

is the general solution of the differential equation

$$y'' - 7y' + 10y = 24e^x$$
 on  $(-\infty, \infty)$ .

4. (12 points) Find the general solution of the given higher order differential equation

$$y''' - 4y'' - 5y' = 0.$$

**Solution:** As we look for solution of the form  $y(x) = e^{rx}$  then the characteristic equation is

$$r^{3} - 4r^{2} - 5r = 0$$
 or  $r(r^{2} - 4r - 5) = 0$ .

As  $r(r^2 - 4r - 5) = r(r - 5)(r + 1)$ , the characteristic equation has roots  $r_1 = 0$ ,  $r_2 = 5$ ,  $r_3 = -1$ . Hence we have the general solution

$$y(x) = c_1 e^0 + c_2 e^{5x} + c_3 e^{-x}.$$

5. For linear differential equations of the form

$$x^2y'' + axy' + by = 0$$

we look for solution of the form  $y(x) = x^r$ .

(a) (6 points) Find two solutions of the form  $y(x) = x^r$  for the DE

$$x^2y'' + xy' - 16y = 0.$$

**Solution:** Since we are looking for solution of the form  $y(x) = x^r$  then

$$y'(x) = rx^{r-1}$$
 and  $y''(x) = r(r-1)x^{r-2}$ .

Substituting to the differential equation we get

$$0 = x^{2}y'' + xy' - 16y = x^{2}r(r-1)x^{r-2} + xrx^{r-1} - 16x^{r} = r(r-1)x^{r} + rx^{r} - 16x^{r}.$$

From this we get

$$0 = r(r-1) + r - 16 = r^2 - r + r - 16 = r^2 - 16$$

Hence we have two roots,  $r_1 = -4$  and  $r_2 = 4$ . Hence there are solutions of the form  $y_1(x) = x^{-4} = 1$  and  $y_2(x) = x^4$ .

(b) (3 points) Use part (a) and write down the general solution of the DE.

Solution: Since we got two solutions the general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 x^{-4} + c_2 x^4.$$

(c) (6 points) Using the general solution you found in part (a), find the particular solution satisfying the initial values

$$y(1) = 0$$
 and  $y'(1) = 1$ .

**Solution:** Since  $y(x) = c_1 x^{-4} + c_2 x^4$  we get

$$y(1) = c_1 + c_2 = 0$$
 hence  $c_1 = -c_2$ .

Hence  $y(x) = c_1 x^{-4} - c_1 x^4$ . Using the second initial condition we get

$$y'(x) = -4c_1x^{-4} - 4c_1x^4$$
 and at  $x = 1$   $y'(1) = -8c_1 = 1$ 

we get  $c_1 = -1/8$ . Hence the particular solution to the above initial value problem is

$$y(x) = -\frac{1}{8}x^{-4} + \frac{1}{8}x^4$$

6. (15 points) Find a differential equation that has

$$y(x) = c_1 e^{2x} \cos(3x) + c_2 e^{2x} \sin(3x) + \frac{1}{2} \sin(3x)$$

as its general solution.

Solution: We know

$$y(x) = y_c(x) + y_p(x) = c_1 e^{2x} \cos(3x) + c_2 e^{2x} \sin(3x) + \frac{1}{2} \sin(3x).$$

We first focus on the complementary solution part. We see from

$$y_c(x) = c_1 e^{2x} \cos(3x) + c_2 e^{2x} \sin(3x)$$

that  $r_1 = 2 + 3\mathbf{i}$  and  $r_2 = 2 - 3\mathbf{i}$  and they are not repeated. Hence, the characteristic equation is

$$(r-2) = \pm 3\mathbf{i}$$
 or  $(r-2)^2 = 9\mathbf{i}^2 = -9$ .

From this we get

$$r^2 - 4r + 4 = -9$$
 or  $r^2 - 4r + 13 = 0$ 

is the characteristic equation. We want to find a linear constant coefficient homogeneous differential equation whose characteristic equation is  $r^2 - 4r + 13 = 0$ . The DE would be

$$y'' - 4y' + 13y = 0.$$

We now return back to the  $y_p(x) = \frac{1}{2}\sin(3x)$  solution. Now we will find the right.

$$y_p'' - 4y_p' - 5y_p = -\frac{9}{2}\sin(3x) - 6\cos(3x) + \frac{13}{2}\sin(3x) = 2\sin(3x) - 6\cos(3x).$$

Hence combining these two we get  $y(x) = c_1 e^{2x} \cos(3x) + c_2 e^{2x} \sin(3x) + \frac{1}{2} \sin(3x)$  is a solution of

$$y'' - 4y' - 5y = 2\sin(3x) - 6\cos(3x).$$

7. (15 points) Find the solution to the initial value problem

$$y'' - y = xe^x$$
 with  $y(0) = 1$  and  $y'(0) = -\frac{1}{4}$ .

**Solution:** We first solve the homogeneous equation y'' - y = 0. To this end, we find the characteristic equation

$$r^{2} - 1 = 0$$
 and  $(r - 1)(r + 1) = 0$ .

From this we get  $r_1 = 1$  and  $r_1 = -1$ . Since they are not repeated we get

$$y_c(x) = c_1 e^x + c_2 e^{-x}$$

is the complementary solution. We now focus on the particular solution. Since the right hand side contains  $xe^x$  and the complementary solution contains  $e^x$  then we should consider

$$y_p(x) = Axe^x + Bx^2e^x.$$

To find A and B we use the fact the  $y_p$  is a solution to

$$y_p'' - y_p = xe^x$$

To this end,

$$y'_{n}(x) = Ae^{x} + Axe^{x} + 2Bxe^{x} + Bx^{2}e^{x}$$

and

$$y_p''(x) = Ae^x + Ae^x + Axe^x + 2Be^x + 2Bxe^x + 2Bxe^x + Bx^2e^x$$

Using this in the DE we get

$$Ae^{x} + Ae^{x} + Axe^{x} + 2Be^{x} + 2Bxe^{x} + 2Bxe^{x} + Bx^{2}e^{x} - Axe^{x} - Bx^{2}e^{x} = xe^{x}$$

Combining  $xe^x$  and  $e^x$  we see that

$$4Bxe^x + (2A + 2B)e^x = xe^x$$

which gives us 4B = 1 or B = 1/4 and 2A + 2B = 0 which gives us A = -1/4. Hence

$$y(x) = y_c(x) + y_p(x) = c_1 e^x + c_2 e^{-x} - \frac{1}{4} x e^x + \frac{1}{4} x^2 e^x$$

is the general solution. Now we use the initial values y(0) = 1 we see that

$$1 = y(0) = c_1 + c_2$$

and y'(0) = -1/4 which gives us

$$-\frac{1}{4} = y'(0) = c_1 - c_2 - \frac{1}{4}$$

Solving for  $c_1$  and  $c_2$  we get  $c_1 = c_2 = 1/2$ . Hence

$$y(x) = y_c(x) + y_p(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x} + \frac{1}{4}xe^x - \frac{1}{4}x^2e^x$$