Fall 2018 - Math 2410
Practice Exam 3 - December 6
Time Limit: 75 Minutes

Name (Print): ______________________________

This exam contains 8 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	28	
2	12	
3	12	
4	16	
5	16	
6	16	
Total:	100	

1. In parts (a) and (b) find the Laplace transform of each given function.

   (a) (6 points) Given $f(t) = t^2 e^{3t}$. Find $F(s) = \mathcal{L}\{f(t)\}$.

Solution: Using translation property we get

$$\mathcal{L}\{t^2 e^{3t}\} = \mathcal{L}\{t^2\}|_{s\to s-3} = \left.\frac{2!}{s^3}\right|_{s\to s-3} = \frac{2}{(s-3)^3}.$$

   (b) (8 points) Given

$$f(t) = \begin{cases} 2t & \text{when } 0 \le t \le 1, \\ 1 & \text{when } 1 < t < \infty. \end{cases}$$

Find $F(s) = \mathcal{L}\{f(t)\}$.

Solution: Since this is a piecewise function we use the definition of the Laplace transform

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} f(t)dt = \int_0^1 e^{-st} f(t)dt + \int_1^\infty e^{-st} f(t)dt \\ &= \int_0^1 e^{-st} 2t dt + \int_1^\infty e^{-st} 1 dt \\ &= \frac{1}{-s} e^{-st} 2t|_{t=0}^{t=1} + \frac{1}{s}\int_0^1 e^{-st} 2dt + \frac{e^{-st}}{-s}|_{t=1}^{t=\infty} \\ &= \frac{-2}{s} e^{-s} - \frac{2}{s^2} e^{-st}|_{t=0}^{t=1} + \frac{e^{-s}}{s} \\ &= \frac{-2}{s} e^{-s} - \frac{2}{s^2} e^{-s} + \frac{2}{s^2} + \frac{e^{-s}}{s}. \end{aligned}$$

(c) (6 points) Find the integral

$$\int_0^\infty e^{-st}\cos(t)dt \quad (s > 0).$$

(Hint: This integral represents the Laplace transform of a certain function. It is absolutely not necessary to integrate in order to find the answer.)

Solution: Notice that the integral

$$\int_0^\infty e^{-st}\cos(t)dt$$

is the Laplace transform of $f(t) = \cos(t)$. Hence

$$\int_0^\infty e^{-st}\cos(t)dt = \mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1}.$$

(d) (8 points) Find $F(s) = \mathcal{L}\{\cos(t)\mathcal{U}(t-\pi)\}$ where

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a, \\ 1, & t \ge a. \end{cases}$$

Solution: We use the second translation theorem to get

$$F(s) = \mathcal{L}\{\cos(t)\mathcal{U}(t-\pi)\} = e^{-\pi s}\mathcal{L}\{\cos(t+\pi)\} = e^{-\pi s}\mathcal{L}\{-\cos(t)\} = e^{-\pi s}\frac{-s}{s^2+1}.$$

2. (a) (6 points) Given

$$F(s) = \frac{s^2+9}{9s-s^3}.$$

Find $\mathcal{L}^{-1}\{F(s)\}$.

Solution: We should first do partial fraction;

$$\frac{s^2+9}{9s-s^3} = \frac{A}{s} + \frac{B}{3-s} + \frac{C}{3+s}.$$

After some calculations you should find $A = 1$, $B = 1$, and $C = -1$. Hence

$$\frac{s^2+9}{9s-s^3} = \frac{1}{s} + \frac{1}{3-s} - \frac{1}{3+s}.$$

Then

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\{\frac{s^2+9}{9s-s^3}\} = \mathcal{L}^{-1}\{\frac{1}{s} - \frac{1}{s-3} - \frac{1}{s+3}\} \\ &= \mathcal{L}^{-1}\{\frac{1}{s}\} + \mathcal{L}^{-1}\{\frac{1}{3-s}\} - \mathcal{L}^{-1}\{\frac{1}{3+s}\} \\ &= 1 - e^{3t} - e^{-3t}\end{aligned}$$

(b) (6 points) Given

$$F(s) = \frac{1}{s-4}e^{-2s}.$$

Find $\mathcal{L}^{-1}\{F(s)\}$.

Solution: We use the second translation formula to get

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{\frac{1}{s-4}e^{-2s}\} = e^{4(t-2)}\mathcal{U}(t-2).$$

3. Suppose that $\mathcal{L}\{f(t)\} = \frac{1}{s^2}$ for some function $f(t)$ with given that $f(0) = -1$, $f'(0) = 2$, and $f''(0) = 1$.

(a) (6 points) Find $\mathcal{L}\{tf(t)\}$.

Solution: (The information $f(0) = -1$, $f'(0) = 2$, and $f''(0) = 1$ is not relevant, mistakenly given in the statement.)
We use the formula given for the derivative of transformations to get

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}\mathcal{L}\{f(t)\} = -\frac{-2}{s^3} = \frac{2}{s^3}.$$

(b) (6 points) Find $\mathcal{L}\{t^2 f(t)\}$.

Solution: Similarly, we use the formula given for the derivative of transformations to get

$$\mathcal{L}\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2}\mathcal{L}\{f(t)\} = \frac{3!}{s^4} = \frac{3!}{s^4}.$$

4. (16 points) Using Laplace transform method solve the following DE

$$y'' - 2y' + y = e^{-x} \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 0.$$

Solution: If we take the Laplace transform of both sides we get

$$\mathcal{L}\{y'' - 2y' + y\} = \mathcal{L}\{e^{-x}\} \quad \text{or} \quad \mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{e^{-x}\}$$

where we have used the linearity property. Now the using Laplace transform of derivatives we get

$$s^2\mathcal{L}\{y(x)\} - sy(0) - y'(0) - 2s\mathcal{L}\{y(x)\} + 2y(0) + \mathcal{L}\{y(x)\} = \frac{1}{s+1}.$$

Solving for $\mathcal{L}\{y(x)\}$ we get

$$(s^2 - 2s + 1)\mathcal{L}\{y(x)\} = \frac{1}{s+1} \quad \text{or} \quad \mathcal{L}\{y(x)\} = \frac{1}{s+1}\frac{1}{(s-1)^2}.$$

Since

$$\frac{1}{s+1}\frac{1}{(s-1)^2} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

we need to find A, B, C. After some algebra, one can find $= 1/4, B = -1/4, C = 1/2$ and hence we have

$$\mathcal{L}\{y(x)\} = \frac{1}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2}.$$

Using the inverse Laplace transform on both sides we get

$$\mathcal{L}^{-1}\mathcal{L}\{y(x))\} = \mathcal{L}^{-1}\{\frac{1}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2}\}.$$

Hence using linearity we have

$$y(x) = \frac{1}{4}\mathcal{L}^{-1}\{\frac{1}{s+1}\} - \frac{1}{4}\mathcal{L}^{-1}\{\frac{1}{s-1}\} + \frac{1}{2}\mathcal{L}^{-1}\{\frac{1}{(s-1)^2}\}\}.$$

This gives us

$$y(x) = \frac{1}{4}e^{-x} - \frac{1}{4}e^{x} + \frac{1}{2}xe^{x}.$$

5. (16 points) Using Laplace transform method solve the following DE

$$y' + y = f(t) \quad \text{with} \quad y(0) = 5 \quad \text{where} \quad f(t) = \begin{cases} 0 & 0 \le t < \pi, \\ 3\cos(t), & t \ge \pi. \end{cases}$$

Solution: Notice first that $f(t) = 3\cos(t)\mathcal{U}(t-\pi)$. We now take the Laplace transform of both sides to get

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$
$$sY(s) - y(0) + Y(s) = -3\frac{s}{s^2+1}e^{-\pi s}.$$

From this and using the given initial value we get

$$(s+1)Y(s) = 5 - 3\frac{s}{s^2+1}e^{-\pi s}.$$

Hence

$$\begin{aligned} Y(s) &= \frac{5}{s+1} - 3e^{-\pi s}\frac{1}{(s^2+1)(s+1)} \\ &= \frac{5}{s+1} - 3e^{-\pi s}[\frac{A}{s+1} + \frac{Bs+C}{s^2+1}] \\ &= \frac{5}{s+1} - \frac{3}{2}e^{-\pi s}[\frac{1}{s+1} + \frac{-s+1}{s^2+1}]. \end{aligned}$$

At this point we take the inverse Laplace transform of both sides and use the linearity to get

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\} &= \mathcal{L}^{-1}\{\frac{5}{s+1} - \frac{3}{2}e^{-\pi s}[\frac{1}{s+1} + \frac{-s+1}{s^2+1}]\} \\ &= \mathcal{L}^{-1}\{\frac{5}{s+1}\} - \frac{3}{2}\mathcal{L}^{-1}\{e^{-\pi s}[\frac{1}{s+1} + \frac{-s+1}{s^2+1}]\} \\ &= 5e^{-t} - \frac{3}{2}\mathcal{L}^{-1}\{e^{-\pi s}\frac{1}{s+1}\} + \frac{3}{2}\mathcal{L}^{-1}\{e^{-\pi s}\frac{s}{s^2+1}\} - \frac{3}{2}\mathcal{L}^{-1}\{e^{-\pi s}\frac{1}{s^2+1}\} \\ &= 5e^{-t} - \frac{3}{2}e^{-(t-\pi)}\mathcal{U}(t-\pi) + \frac{3}{2}\cos(t-\pi)\mathcal{U}(t-\pi) - \frac{3}{2}\sin(t-\pi)\mathcal{U}(t-\pi) \\ &= 5e^{-t} - \frac{3}{2}e^{-(t-\pi)}\mathcal{U}(t-\pi) - \frac{3}{2}\cos(t)\mathcal{U}(t-\pi) + \frac{3}{2}\sin(t)\mathcal{U}(t-\pi). \end{aligned}$$

6. **LRC-Series Circuits**. The charge on the capacitor is related to the current $i(t)$ by $i = dq/dt$ which satisfies
$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t).$$
Let $L = 1h$, $R = 100\Omega$, $C = 0.0004f$, $E(t) = 40V$, $q(0) = 0C$, $i(0) = 5A$.

(a) (6 points) Find the charge $q(t)$

Solution: Since $L = 1h$, $R = 100\Omega$, $C = 0.0004f$, $E(t) = 40V$, $q(0) = 0C$, $i(0) = 5A$ we have
$$\frac{d^2q}{dt^2} + 100\frac{dq}{dt} + \frac{1}{0.0004}q = 40.$$
We need to solve the DE to find $q(t)$. This is a constant coefficient and we first find the characteristic equation
$$r^2 + 100r + 2500 = 0$$
From we see that $r = -50$ with double root. From this we see that the complementary solution is
$$q(t) = c_1e^{-50t} + c_2te^{-50t}.$$
One can see that $q_p(t) = 2/125$ (you can test $q_p(t) = c$ and then find c). Hence
$$q(t) = q_c(t) + q_p(t) = c_1e^{-50t} + c_2te^{-50t} + \frac{2}{125}.$$
Since $q(0) = 0$ we see that $c_1 = -2/125$. Since $i(0) = q'(0) = 0$ we get
$$5 = \frac{100}{125} + c_2 \quad \text{i.e.} \quad c_2 = 5 - \frac{100}{125} = \frac{21}{5}.$$
Hence
$$q(t) = -\frac{2}{125}e^{-50t} + \frac{21}{5}te^{-50t} + \frac{2}{125}.$$

(b) (6 points) Find the current $i(t)$.

Solution: Since $i(t) = q'(t)$ we get
$$i(t) = \frac{100}{125}e^{-50t} + \frac{21}{5}e^{-50t} - 210te^{-50t} = e^{-50t} - 210te^{-50t}$$

(c) (4 points) Find the maximum charge on the capacitor.

Solution: This happens when $i(t) = q'(t) = 0$.
$$i(t) = 0 = e^{-50t} - 210te^{-50t}.$$
which gives $t = 1/210$ and the maximum charge is $q(1/210)$.