Lotka-Volterra Equations

Andrew Premus

Spring 2019

MATH 2410Q – Differential Equations

Lotka-Volterra Equations

The Lotka-Volterra Equations are commonly known as the predator-prey equations. They are a pair of equations, used to model the populations of two species where each population depends on the other.

As the predator population in a given region grows, the prey population decreases, until there is not enough food for the predators. So, the predator population then decreases, and the prey population can grow again.

Contributors

- American biophysicist Alfred J. Lotka initially proposed the model in 1910, in the theory of autocatalytic chemical reactions.

- Italian mathematician Vito Volterra proposed the model in 1926, independently from Lotka, to explain observations of changes in fish population after World War I.

Additional Contributors

- Russian mathematician Andrey Kolmogorov helped Lotka extend the model to “organic systems” using a plant species and a herbivorous animal species, and generalized the predator-prey system in 1936.
- Italian marine biologist Umberto D’Ancona, Volterra’s son-in-law, inspired Volterra to develop the model with respect to his studies of predatory fish populations during and after World War I.

Andrey Nikolaevich Kolmogorov

Equations

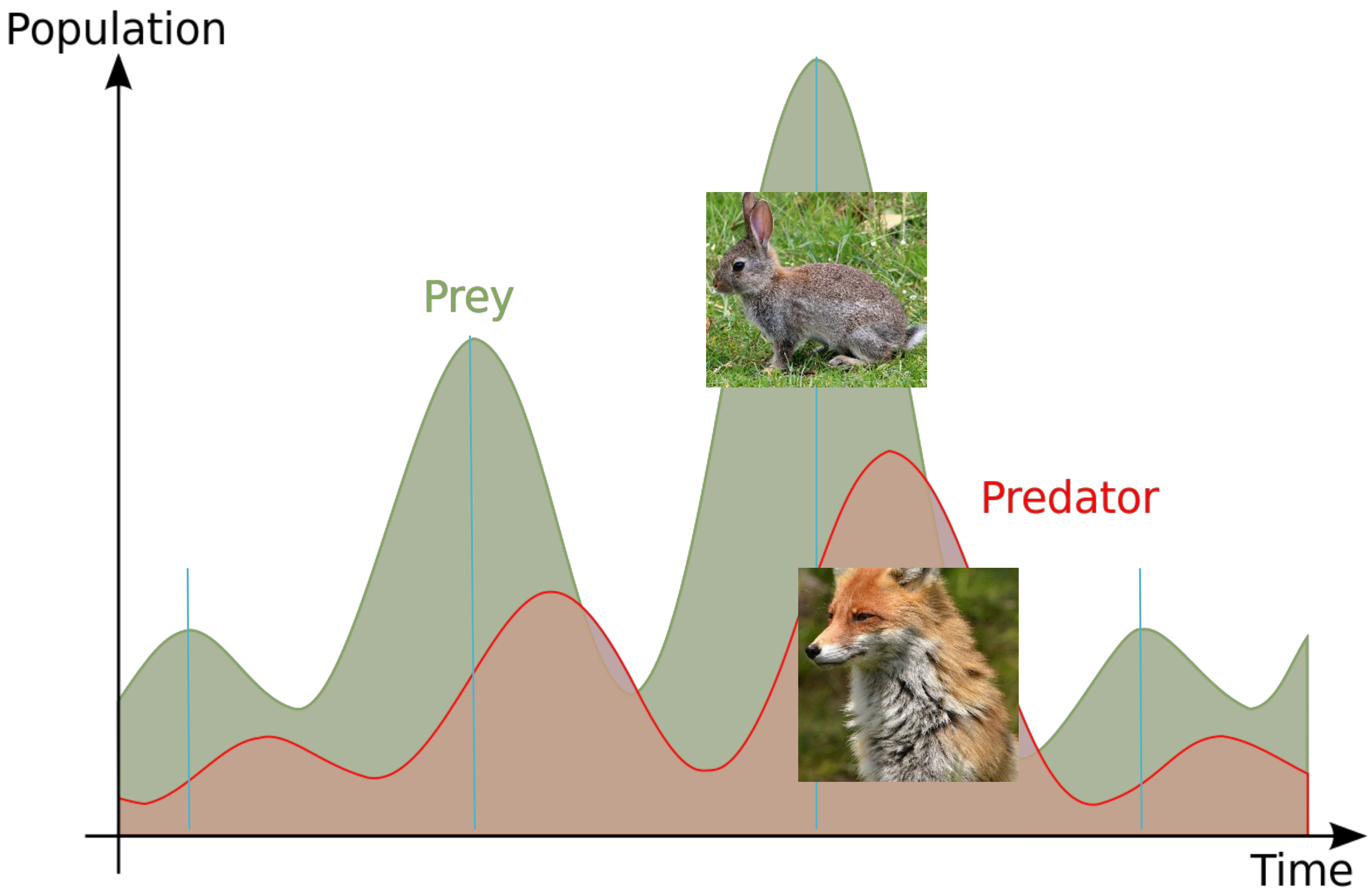
- x is the number of prey
- y is the number of some predator
- dy/dt and dx/dt represent the instantaneous growth rates of the two populations
- t represents time
- α, β, γ, and δ are positive, real coefficients describing the interaction of x and y

Parameter Coefficients

$x = x(t)$ Number of Prey
$y = y(t)$ Number of Predators

$\alpha > 0$ Birth Rate of Prey (x)
$\beta > 0$ Death Rate of Prey/Predator (xy)
$\gamma > 0$ Death Rate of Predators (y)
$\delta > 0$ Birth Rate of Predators/Prey (xy)

Knowns and Unknowns

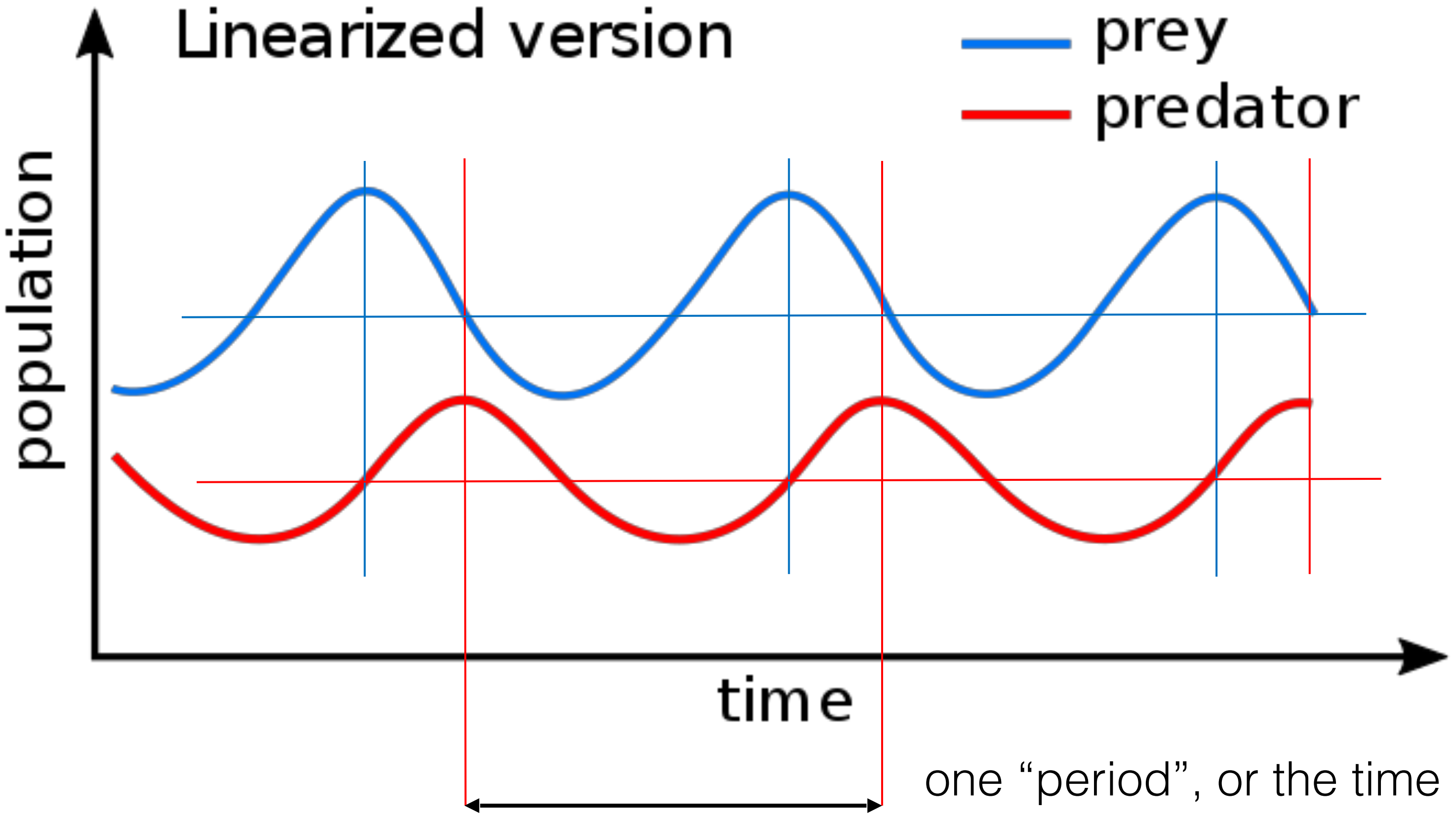


The coefficients α, β, γ, and δ are known constants.

If the populations of x and y at time t are known, then the rates of change of the populations can be calculated algebraically.

If both rates at time t are known, then both populations can be found at this time t using a matrix (two equations, two unknowns).

Linearized Solution of the Non-Linear DE



Linearization yields a solution that resembles SHM

The prey population leads that of the predators by 90° in the cycle.

one “period”, or the time it takes for the population to return to equilibrium value

Linearization Solution (Extinction)

Given this set of equations,

the solutions can be found by taking the partial derivatives of each equation with respect to x and y (both populations).

Linearization Solution

These partial derivatives come out to be:

This is evaluated at the first fixed point, which is also called the extinction point. This is when both populations are eventually zero.

Linearization Solution

Taking the determinant of these four partial derivatives at the equilibrium point (0,0) gives:

$\alpha > 0$ Birth Rate of Prey
$\gamma > 0$ Death Rate of Predators

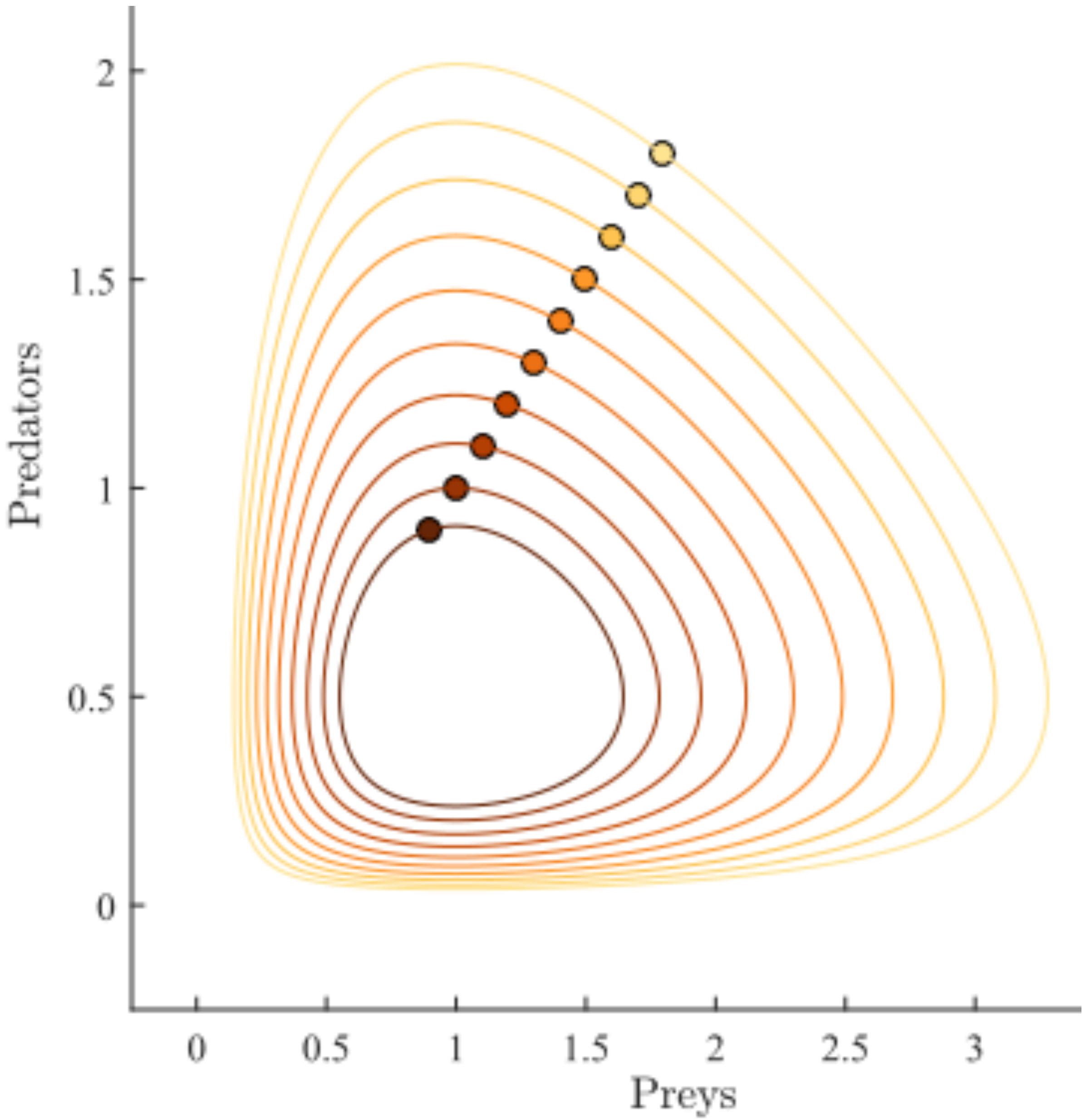
This solution is considered the extinction solution, because both of the populations are zero, and are therefore unable to change.

Oscillation Solution

Evaluating the Jacobian matrix again, but at the second fixed point:

The solutions to this matrix are purely imaginary and conjugate to each others

The fixed point is elliptic so the solutions are periodic as shown next.



Predators
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
Preys

Stability

- In order to create a stable ecosystem, extinction events like this must be avoided.
- The matrix must have all positive eigenvalues –the determinant of the matrix must have a non zero population for both species.
- If the prey species goes extinct, then the predator species will follow with time.
- If the predator species goes extinct, then the prey population can grow without bound.
- The populations of prey and predator can get very close to zero and still recover.

Applications of Lotka-Volterra Equations

- Predicting populations of competing species in any ecosystem
- Analyzing autocatalytic chemical reactions
- Competing products on the market, in terms of availability

References

Lotka–Volterra equations. (2018, October 03). Retrieved from https://en.wikipedia.org/wiki/Lotka–Volterra_equations

Competitive Lotka–Volterra equations. (2018, September 21). Retrieved from https://en.wikipedia.org/wiki/Competitive_Lotka–Volterra_equations

Sprinkle, J. (2013, September 12). Linearizing at an equilibrium point (Lotka-Volterra). Retrieved from https://www.youtube.com/watch?v=k_IkbxwSK7g