UCONN - Math 3150 - Fall 2018 - Problem set 1

Question 1 (1.4, Page 5) *a. Guess a formula for* $1 + 3 + \ldots + (2n - 1)$ *by evaluating the sum for* $n = 1, 2, 3, 4$.

b. Prove your formula using mathematical induction.

Solution:

(a) For $n = 1$ then the sum is just 1.
For $n = 2$, then we have $1 + (2.2 - 1) = 1 + 3 = 4$.
For $n = 3$, then we have $1 + 3 + 5 = 9$.
For $n = 4$, then we have $1 + 3 + 5 + 7 = 16$.

(b) It looks like for $n = 1$ sum is 1. For $n = 2$ the sum is 4 and it looks like $2^2 = 4$ and for 3 the sum is 9. Therefore we claim

$$(P_n) \quad \text{“The sum of } 1 + 3 + \ldots + (2n - 1) \text{ is } n^2\text{”}.$$

Using mathematical induction we shall prove this.

For $n = 1$ we have the base step;

$$(P_1) \quad \text{sum of 1 is 1.}$$

It is clearly true. We see that the base step (P_1) holds. We now assume that (P_n) holds and we will show that $(P_{n+1}$ holds as well. Consider

$$\begin{aligned} 1 + 3 + \ldots + (2n - 1) + (2(n + 1) - 1 &= 1 + 3 + \ldots + (2n - 1) + 2n + 1 \\ &= n^2 + 2n + 1 = (n + 1)^2. \end{aligned}$$

This shows us the sum $1 + 3 + \ldots + (2n - 1) + (2(n + 1) - 1$ is $(n + 1)^2$ which shows (P_{n+1}) is true. By mathematical induction we conclude that our statement is correct. Hence

$$1 + 3 + \ldots + (2n - 1) = n^2.$$

Question 2 (1.7, Page 5) *Prove* $7^n - 6n - 1$ *is divisible by* 36 *for all positive integers* n.

Solution: Now our statement is

$$(P_n) \quad \text{“}7^n - 6n - 1 \text{ is divisible by 36 for } n = 1, 2, \ldots\text{”}.$$

We start with checking the base step, i.e., when $n = 1$. In this case we have $7^n - 6n - 1 = 0$ and it is clear that 0 is divisible by 36. Hence the base step P_1 is true. We next assume that P_n holds and want to prove that P_{n+1} is also true using P_n. As $(P_{n+1}$ is $7(n + 1)^2 - 6(n + 1) - 1$ and we consider

$$\begin{aligned} 7^{n+1} - 6(n + 1) - 1 &= 7^{n+1} - 6n - 6 - 1 \\ &= 7(7^n - 1) - 6n \\ &= 7(7^n - 6n - 1 - 6n + 42n \\ &= 7(7^n - 6n - 1) - 36n. \end{aligned}$$

Observe that the first part is $(7^n - 6n - 1)$ which is the induction hypothesis (i.e., (P_n)) and we know it is true. Hence it is divisible by 36. The remaining part i.e., $-36n$ is obviously divisible by 36. Therefore, $7(7^n - 6n - 1) - 36n$ is divisible by 36. By mathematical induction we conclude that (P_n) is true;

$$7^n - 6n - 1 \text{ is divisible by 36 for } n = 1, 2, \ldots.$$

Question 3 (1.11, Page 5) *For each $n \in \mathbb{N}$, let P_n denote the assertion "$n^2 + 5n + 1$ is an even integer".*

a. Prove P_{n+1} is true whenever P_n is true.

b. For which n is P_n actually true? What is the moral of the exercise?

Solution:

(a) If we assume P_n is true and consider P_{n+1} we have

$$(n+1)^2 + 5(n+1) + 1 = n^2 + 2n + 1 + 5n + 5 + 1 = n^2 + 5n + 1 + 2n + 6.$$

We observe from this that the the red part $n^2 + 5n + 1$ is P_n i.e., induction hypothesis. Therefore it is even integer. The blue part $2n + 6 = 2(n+2)$ also even. Therefore, the whole sum is even. This shows that P_{n+1} is also true.

(b) P_n is NOT true for any integer. In fact one can show that $n^2 + 5n + 1$ is odd integer for every n. The moral here is that we can not skip checking the base step.

Question 4 (2.3, Page 13) *Show $\sqrt{2 + \sqrt{2}}$ is not a rational number.*

Solution: We consider $x = \sqrt{2 + \sqrt{2}}$ or equivalently

$$x^2 = 2 + \sqrt{2} \quad \text{equivalently } quad (x^2 - 2)^2 = 2.$$

Hence we have

$$(x^-2)^2 - 2 = x^4 - 4x^2 - 4 - 2 = x^4 - 4x^2 - 6.$$

From Corollary 2.3 we know that rational solutions of $x^4 - 4x^2 - 6$ should divide $c_0 = 6$. Those numbers are $\pm 1, \pm 2, \pm 3, \pm 6$ by the Rational Zeros Theorem.
When $x = \pm 1$ we have $1 - 4 - 6 \neq 0$.
When $x = \pm 2$ we have $16 - 16 - 6 \neq 0$.
When $x = \pm 3$ we have $81 - 36 - 6 \neq 0$.
When $x = \pm 6$ we have $6^4 - 46^2 - 6 \neq 0$.

Hence we conclude that $x^4 - 4x^2 - 6$ has no rational solution which in turn gives us $\sqrt{2 + \sqrt{2}}$ is not rational.

Question 5 (2.8, Page 13) *Find all rational solutions of the equation $x^8 - 4x^5 + 13x^3 - 7x + 1$.*

Solution: From Corollary 2.3 we know that the rational solutions of $x^8 - 4x^5 + 13x^3 - 7x + 1$ should divide $c_0 = 1$. In this case, if there are rational solutions they have to be ± 1. For $x = 1$ we have $1 - 4 + 13 - 7 + 1 \neq 0$. For $x = -1$ we have $1 + 4 - 13 + 7 + 1 = 0$. Hence we have $x = -1$ is a solution and it is the only rational solution.

Question 6 (3.6, Page 19) *a. Prove* $|a+b+c| \leq |a|+|b|+|c|$ *for all* a, b, c.

b. Use induction to prove $|a_1+a_2+\ldots+a_n| \leq |a_1|+|a_2|+\ldots+|a_n|$.

Solution: (a) We can use the triangle inequality as follows twice

$$|a+b+c| = |(a+b)+c| \leq |a+b|+|c| \leq |a|+|b|+|c|.$$

This finishes the proof.

(b) We first step our inductions step. If P_n denote the assertion "$|a_1+a_2+\ldots+a_n| \leq |a_1|+|a_2|+\ldots+|a_n|$ we then first show this is true for P_1 and assuming P_n is true then we show P_{n+1} is also true.

P_1 is clearly true as $|a_1| = |a_1|$. Assume that P_n is true and consider P_{n+1}

$$\begin{aligned} |a_1+a_2+\ldots+a_n+a_{n+1} &= |(a_1+a_2+\ldots+a_n)+a_{n+1}| \\ &\leq |(a_1+a_2+\ldots+a_n)|+|a_{n+1}| \\ &\leq |a_1|+|a_2|+\ldots+|a_n|+|a_{n+1} \end{aligned}$$

where we have used triangle inequality in the second line and the induction hypothesis that P_n is true. This finishes the proof.

Question 7 (3.7, Page 7) *1. Show* $|b| < a$ *if and only if* $-a < b < a$.

2. Show $|a-b| < c$ *if and only if* $b-c < a < b+c$.

3. Show $|a-b| \leq c$ *if and only if* $b-c \leq a \leq b+c$.

Solution: (a) Here there are two statements we need to prove. If $|b| < a$ then $-a < b < a$ and we also need to show $-a < b < a$ then $|b| < a$. We first assume $|b| < a$. Since $|b| < a$ then we also have $-a < -|b|$ and since $-|b| \leq 0 \leq |b|$ we have $-a < -|b| \leq |b| < a$. Finally b is either $|b|$ or $-|b|$ we conclude that $-a < b < a$. This finishes the first proof. We now return the second proof. Assume $-a < b < a$. Then if we multiply this by -1 we get $-a < -b < a$ and since $|b|$ is either $-b$ or b we conclude that $-a < |b| < a$.

(b) We use part (a) (replace b by $a-b$ we get $|a-b| < c$ if and only if $-c < a-b < c$. Then adding b to this inequality we get $b-c < a-b+b = a < b+c$.

(c) We can reprove parts (a) with $<$ replaced by $\leq$. We next reprove part (b) with $<$ replaced by $\leq$ which is (c). This finishes the proof of c.

Question 8 (4.14, Page 26) *Let A and B be nonempty bounded subset of R.*

a. Prove $\sup(A+B) = \sup(A)+\sup(B)$.

b. Prove $\inf(A+B) = \inf(A)+\inf(B)$.

Solution: (a) Let $M = \sup A$ and $N = \sup B$. Let $\epsilon > 0$ be given. Since $M-\epsilon/2 < M = \sup A$ we can $a \in A$ such that $M-\epsilon/2 < a$. Similarly, $N-\epsilon/2 < N = \sup B$ we can $b \in B$ such that $N-\epsilon/2 < b$. Now combining these two we get

$$M-\epsilon/2+N-\epsilon/2 < a+b.$$

Note that $a \in A$ and $b \in B$ then by definition $a + b \in A + B$. Therefore, we have

$$M + N - \epsilon < a + b.$$

From this we observe that $M + N - \epsilon < a + b \leq \sup(A + B)$. Since this is true for every $\epsilon > 0$ we see that $M + N \leq \sup(A + B)$.

We proved inequality and we now prove the converse inequality. Let c be an element in $A + B$. Then there is $a \in A$ and $b \in B$ by definition. We also have $a \leq M = \sup A$ and $b \leq N = \sup B$. Combining these two we get $a + b \leq M + N$ (therefore, $M + N$ is an upper bound for $A + B$) and since $\sup(A + B)$ is the least upper bound we see that $\sup(A + B) \leq M + N$. This finishes the converse inequality. Combining these two we have $\sup(A + B) = \sup(A) + \sup(B)$.

(b) One can similarly prove this.