Fall 2018 - Math 3150
Exam 1 - September 27
Time Limit: 75 Minutes

Name (Print): ______________________

This exam contains 7 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	13	
2	10	
3	10	
4	12	
5	10	
6	10	
7	10	
Total:	75	

1. For each of the following statements, say whether it is true or false. If the statement is false, give a counterexample.

   (a) (3 points) The sum of two irrational numbers is irrational.

   Solution: This is a false statement. $\sqrt{2}$ and $1-\sqrt{2}$ are irrational but $\sqrt{2}+1-\sqrt{2}=1$ is rational.

   (b) (4 points) $\sqrt[3]{216}$ is an irrational number.

   Solution: This is a false statement. Since $\sqrt[3]{216}=6$.

   (c) (3 points) Convergent sequences are bounded.

   Solution: This is a true statement as it is proven at Theorem 9.1.

   (d) (3 points) If (s_n) and (t_n) are convergent sequences of real numbers with $\lim s_n = s$ and $\lim t_n = t$. If $s_n < t_n$ for every $n \in \mathbb{N}$ then $s < t$.

   Solution: This is also a false statement. Consider $s_n = 1 - 1/n$ and $t_n = 1 + 1/n$. It is clear that $s_n < t_n$ for every $n \in \mathbb{N}$. But $\lim s_n = 1 = \lim t_n$.

2. (10 points) Let (s_n) be a sequence of real numbers for $n \in \mathbb{N}$.

   Show that $\lim s_n = 0$ if and only if $\lim |s_n| = 0$.

   Solution: See the lecture notes or the book for a proof.

3. (10 points) Prove that $\sqrt{1+\sqrt{2}}$ is irrational.

Solution: Let $x = \sqrt{1+\sqrt{2}}$ then $x^2 = 1+\sqrt{2}$ or $x^2 - 1 = \sqrt{2}$. Then we have $(x^2-1)^2 = 2$. Therefore we have $x^4 - 2x^2 + 1 - 2 = 0$, i.e.

$$x^4 - 2x^2 - 1 = 0.$$

From the corollary of the Rational zeros theorem we know that any solution of this equation must be an integer which divides 1. The only possible rational solutions are ± 1. It is clear that none of these numbers are solutions. Therefore x is not a rational number.

4. Let $S = \{x \text{ irrational};\ 0 \leq x \leq 2\}$.

(a) (4 points) What are $\sup(S)$ and $\inf(S)$?

Solution: $\sup(S) = 2$ and $\inf(S) = 0$.

(b) (4 points) Does the set S have a maximum element and/or a minimum element?

Solution: The set does not have a maximum element as $\sup(S) = 2$ and $2 \notin S$. With the same reasoning, since $\inf(S) = 0$ and $0 \notin S$ the set does not have a minimum element.

(c) (4 points) Write a decreasing sequence (s_n) such that $s_n \in S$ for each $n \in \mathbb{N}$ for which (s_n) converges to $\inf(S)$. Here decreasing means that $s_n \geq s_{n+1}$ for every $n \in \mathbb{N}$.

Solution: We know that $\sqrt{2}$ is not a rational number and dividing it by an integer will not change that property. Therefore if we consider $s_n = \sqrt{2}/n$ for $n \in \mathbb{N}$ we see that $s_n \to 0 = \inf(S)$ with each $s_n \in S$ for every $n \in \mathbb{N}$.

5. (10 points) Let (t_n) and (s_n) be two sequences of real numbers for which t_n converges to t and s_n converges to s. Prove that $\lim(t_n + s_n) = s + t$.

Solution: See lecture notes or proof of Theorem 9.3 in the book.

6. (10 points) Using ϵ-definition, prove that

$$\lim \frac{2n-1}{n^2+1} = 0.$$

Solution: Given $\epsilon > 0$ we want to find integer N large enough so that if $n > N$ we always have

$$\left|\frac{2n-1}{n^2+1} - 0\right| < \epsilon.$$

Let us focus on $\frac{2n-1}{n^2+1}$ a bit. We want to estimate it from above. Therefore, we need to estimate numerator from above and denominator from below. It is clear that $2n - 1 \leq 2n$ for all $n \in \mathbb{N}$. Also, $n^2 + 1 \geq n^2$ for every $n \in \mathbb{N}$. Now

$$\left|\frac{2n-1}{n^2+1} - 0\right| \leq \frac{2n}{n^2} \leq \frac{2}{n}$$

and we want this to be smaller than ϵ. i.e., $2/n < \epsilon$ equivalently, $2/\epsilon < n$. This suggests us that we can choose $N = 2/\epsilon$.

Formal Proof: Let $\epsilon > 0$ be given and choose $N = 2/\epsilon$. Then if $n > N = 2/\epsilon$ we have

$$\epsilon > \frac{2}{n} = \frac{2n}{n^2} \geq \left|\frac{2n-1}{n^2+1} - 0\right|.$$

This proves that

$$\lim \frac{2n-1}{n^2+1} = 0.$$

7. (10 points) Using the definition 9.8, (given $M > 0$, there is a number N such that $n > N$ implies $s_n > M$) prove that

$$\lim \frac{n^3+3150}{n+2018} = \infty.$$

Solution: Given $M > 0$ we need to decide N such that if $n > N$ then we want to have

$$\frac{n^3+3150}{n+2018} > M.$$

Let us take a look at $\frac{n^3+3150}{n+2018}$ carefully. We want to estimate this from below in order to choose N. Now $n^3 + 2018 > n^3$ for every $n \in \mathbb{N}$. Also, $n + 2018 \leq n + 2018n = 2019n$. Then combining these we get

$$\frac{n^3+3150}{n+2018} \geq \frac{n^3}{2019n} = \frac{n^2}{2019}$$

and we want M to be smaller than this, i.e. $\frac{n^2}{2019} > M$. If we solve for n here we get

$n > (2019M)^{1/2} = N$.

Formal Proof: Let $M > 0$ be given. Choose $N = (2019M)^{1/2}$. If $n > N = (2019M)^{1/2}$ then we have $n^2 > 2019M$ equivalently,

$$M < \frac{n^2}{2019} = \frac{n^3}{2019n} \leq \frac{n^3 + 3150}{n + 2018}.$$

This show us that

$$\lim \frac{n^3 + 3150}{n + 2018} = \infty.$$