Fall 2018 - Math 3150
Exam 2 - October 30
Time Limit: 75 Minutes

Name (Print): ____________________

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	12	
2	12	
3	19	
4	10	
5	12	
6	10	
7	0	
Total:	75	

1. For each of the following statements, say whether it is true or false. If the statement is false, give a counterexample.

   (a) (4 points) For all sequences of real numbers (s_n) we have $\liminf s_n \leq \limsup s_n$.

   Solution: This is a true statement.

   (b) (4 points) Every monotone sequence of real numbers is convergent.

   Solution: False. Consider $a_n = n$. This is an increasing sequence but it is not convergent.

   (c) (4 points) Every bounded sequence of real numbers has at least one convergent subsequence.

   Solution: True. This is a theorem (Bolzano-Weierstrass Theorem).

2. If possible, give an example of each of the following. Write "not possible" when appropriate.

(a) (4 points) A sequence (s_n) with $\limsup s_n = \infty$ and $\liminf s_n = 0$.

Solution: Consider

$$a_n = \begin{cases} \frac{1}{n} & \text{when } n \text{ is even,} \\ n & \text{when } n \text{ is odd,} \end{cases}$$

Clearly, even terms are converging to $0 = \liminf a_n$ and odd terms are converging to ∞ hence $\limsup a_n = \infty$.

(b) (4 points) A bounded sequence which is not convergent.

Solution: $a_n = (-1)^n$. This is clearly a bounded sequence which is not convergent.

(c) (4 points) Give an example of a bounded sequence of real numbers with exactly two subsequential limits.

Solution: $a_n = (-1)^n$. Even terms are converging to 1 and odd terms are converging to 0.

3. Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

(a) (3 points) Find s_2, s_3, and s_4.

Solution: $s_2 = 2/3$, $s_3 = 5/9$, $s_4 = 14/27$.

(b) (4 points) Use induction to show $s_n > 1/2$ for all $n \in \mathbb{N}$.

Solution: The base case, $n = 1$, we trivially have it. As $s_1 = 1 > 1/2$. Assume that $s_n > 1/2$ and consider $n + 1$;

$$s_{n+1} = \frac{1}{3}(s_n + 1) > \frac{1}{3}(\frac{1}{2} + 1) = \frac{1}{2}.$$

Hence $s_{n+1} > 1/2$. By mathematical induction we conclude that $s_n > 1/2$ for every $n \in \mathbb{N}$.

(c) (4 points) Show (s_n) is a decreasing sequence.

Solution: Since $s_n > 1/2$ then $1 < 2s_n$ for every $n \in \mathbb{N}$. Using this we get

$$s_{n+1} = \frac{1}{3}(s_n + 1) \leq \frac{1}{3}(s_n + 2s_n) = \frac{3s_n}{3} = s_n.$$

We just proved that $s_{n+1} \leq s_n$ for every $n \in \mathbb{N}$. This finishes the proof.

(d) (4 points) Show $\lim s_n$ exists and find $\lim s_n$.

Solution: Since s_n is decreasing sequence from part (c) and bounded from below by part (b) we conclude by theorem that we proved in class that s_n converges. Let $\lim s_n = s$. Then

$$s = \lim s_{n+1} = \lim \frac{1}{3}(s_n + 1) = \frac{1}{3}(\lim s_n + 1) = \frac{1}{3}(s + 1).$$

From this we see that $3s = s + 1$ or $s = 1/2$.

(e) (4 points) Is (s_n) a Cauchy sequence?

Solution: Since (s_n) is a convergent sequence therefore (s_n) is a Cauchy sequence.

4. (10 points) Let (s_n) be any sequence. There exists a monotonic subsequence whose limit is $\liminf s_n$.

Solution: This is Theorem 11.7.

5. Let $S = \{1/n : n \in \mathbb{N}\}$.

   (a) (4 points) Prove that S is not closed.

   Solution: Since $(s_n) = 1/n$ is a sequence with $s_n \in S$ with $\lim s_n = 0$. Since $0 \notin S$ we conclude that S is not closed.

   (b) (4 points) Prove that S is not open.

   Solution: If it was open then for every $s \in S$ and there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset S$. For fix n and $\epsilon > 0$ the set

$$(\frac{1}{n} - \epsilon, \frac{1}{n} + \epsilon)$$

   contains an irrational element, say a_0. Now a_0 can not be in S as it does not contain any irrational elements. Hence S can not be open.

   (c) (4 points) Prove that $S \cup \{0\}$ is compact.

   Solution: We need to show that $S \cup \{0\}$ is closed and bounded (using a theorem we proved in the class). This set is bounded above by 1 and below by 0. Hence it is bounded. Now we will show that it is a closed set. 0 is the only limit of $1/n$ and therefore $S \cup \{0\}$ contains all the limit sets as well. Therefore, it is a closed set. Hence $S \cup \{0\}$ is compact.

6. Let f be a function on $[0,1]$ defined by

$$f(x) = \begin{cases} x & \text{when } x \in [0,1] \text{ rational,} \\ 0 & \text{when } x \in [0,1] \text{ irrational.} \end{cases}$$

(a) (5 points) Use $\epsilon - \delta$ definition to show that f is continuous at $x = 0$.

Solution: Let $\epsilon > 0$ be given. We want to find δ (probably in terms of ϵ) so that whenever $|x - 0| < \delta$ and $x \in [0,1]$ then we necessarily have

$$|f(x) - f(0)| < \epsilon.$$

Note that if $x \in [0,1]$ rational then $f(x) = x$ and hence $|f(x) - f(0)| = |x - 0|$ and we know that $|x - 0| < \delta$. We see that it is enough to choose $\delta = \epsilon$. If $x \in [0,1]$ irrational then $f(x) = 0$ and $||f(x) - f(0)| = |0 - 0| = 0 < \delta$ and therefore it is enough to choose $\delta = \epsilon$ again.

Formal proof: Given $\epsilon > 0$, choose $\delta = \epsilon$. Then if $x \in [0,1]$, $|x - 0| < \delta$ and x is rational then

$$|f(x) - f(0)| = |x - 0| < \delta = \epsilon.$$

If $x \in [0,1]$, $|x - 0| < \delta$ and x is irrational then

$$|f(x) - f(0)| = |0 - 0| < \delta = \epsilon.$$

Hence f is continuous at $x = 0$.

(b) (5 points) Use $\epsilon - \delta$ definition to show that f is discontinuous at all other rational points in $(0,1]$.

Solution: Let x_0 be a fixed rational point in $(0,1]$. Let $\epsilon = \frac{x_0}{2} > 0$. Then for every $\delta > 0$ then the set $(x_0 - \delta, x_0 + \delta)$ contains an irrational element say x_δ (due to density of irrationals). Then $|x_0 - x_\delta| < \delta$ but

$$|f(x_\delta) - f(x_0)| = |0 - x_0| > \frac{x_0}{2} = \epsilon.$$

Hence f is not continuous at any other rational points in $(0,1]$.

7. Let (s_n) be a bounded sequnece of real numbers.

(a) (4 points (bonus)) Carefully state the definition of $\limsup s_n$ and $\liminf s_n$.

Solution: For each $N \in \mathbb{N}$ let

$$u_N = \inf\{s_N, s_{N+1}, s_{N+2}, \ldots,\} \quad \text{and} \quad v_N = \sup\{s_N, s_{N+1}, s_{N+2}, \ldots,\}.$$

Note that (u_N) is an increasing sequence and also it is bounded as (s_n) is bounded then we know that bounded monotone sequences are bounded. Therefore, (u_N) converges to a number say u which is $\liminf s_n$. On the other hand, (v_N) is a decreasing sequence and similarly it is also bounded because of the same reason. We conclude that (v_N) also converges to a number say v which is $\limsup s_n$.

(b) (4 points (bonus)) If $s_n = (-1)^n$, find $\limsup s_n$ and $\liminf s_n$.

Solution: Since the subsequential limit set of (s_n) is $\{-1, 1\}$. We know that $\limsup s_n$ is the largest subsequential limit of (s_n) which is 1 and similarly $\liminf s_n$ is the smallest subsequential limit of (s_n) which is -1. Hence

$$\liminf s_n = -1 \quad \text{and} \quad \limsup s_n = 1.$$

(c) (4 points (bonus)) If $t_n = \sin(n\pi/2)$ then find all subsequential limits of $(t_n s_n)$.

Solution: We already know that (s_n) has two subsequential limits $\{-1, 1\}$, 1 when n is even and -1 when n is odd. We now focus on t_n

$$t_n = \begin{cases} 1 & \text{when} n = 4k+1, \\ 0 & \text{when} n = 4k+2, \\ -1 & \text{when} n = 4k+3, \\ 0 & \text{when} n = 4k. \end{cases}$$

Then if we multiply t_n by s_n we get

$$t_n \cdot s_n = \begin{cases} 1 \cdot (-1) & \text{when} n = 4k+1 = \text{odd}, \\ 0 \cdot 1 & \text{when} n = 4k+2 = \text{even}, \\ -1 \cdot (-1) & \text{when} n = 4k+3 = \text{odd}, \\ 0 \cdot 1 & \text{when} n = 4k = \text{even}. \end{cases}$$

Hence we have $-1, 0, 1$ as subsequential limits. Therefore, the set of subsequential limist of $(t_n \cdot s_n)$ is $\{-1, 0, 1\}$.