UNIVERSITY OF CONNECTICUT

Fall 2018 - Math 3150
Exam 3 - December 6
Time Limit: 75 Minutes

Name (Print): ____________________

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	12	
2	12	
3	12	
4	15	
5	12	
6	12	
7	0	
Total:	75	

1. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & \text{when } x \neq 0, \\ 0 & \text{when } x = 0. \end{cases}$$

(a) (6 points) Show that f is continuous and uniformly continuous on $[-1, 1]$.

Solution: At $x \neq 0$ f is given as a product of the two continuous functions x and $\sin(1/x)$. Also, $\sin(1/x)$ is continuous as it is composition of two continuous functions sin and $1/x$ at points $x \neq 0$. At $x = 0$ we need to show that

$$\lim_{x\to 0} f(x) = f(0) = 0.$$

We will show that $\lim_{x\to 0} |f(x)| = 0$ and this is equivalent of showing what we want. Now

$$|f(x)| = |x\sin(\frac{1}{x})| \leq |x| \to 0 \quad \text{as } x \to 0.$$

Hence

$$\lim_{x\to 0} |f(x)| = 0.$$

This shows that $\lim_{x\to 0} f(x) = 0$ which finishes the proof of continuity.

(b) (6 points) Show that f is not differentiable at $x = 0$.

Solution: Using the definition of derivative we have

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x\sin(\frac{1}{x} - 0}{x - 0} = \lim_{x\to 0} \sin(\frac{1}{x}).$$

We know that $\sin(1/x)$ has no limit as $x \to 0$ and this shows that the above limit does not exit. Hence f is not differentiable at $x = 0$.

2. (12 points) Prove that every continuous function on $[a, b]$ is integrable on $[a, b]$.

Solution: Let f be a continuous function on $[a, b]$. Then f is uniformly continuous on $[a, b]$. Therefore, given $\epsilon > 0$ we can find δ depending only on ϵ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon/(b - a)$.

Now we return back to the proof. Let $\epsilon > 0$ be given and let P be any partitioning of $[a, b]$, $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ with the property that $x - x_{k-1} < \delta$ for every $k = 2, 3, \ldots, n$. Moreover, since f is continuous then f attains its maximum and minimum on every subinterval $[x_{k-1}, x_k]$, say maximum is $M_k = f(z_k)$ on $[x_{k-1}, x_k]$ and minimum is $m_k = f(y_k)$ on $[x_{k-1}, x_k]$. Now since $f(z_k) - f(y_k) = M_k - m_k < \epsilon/(b - a)$ as $y_k, z_k \in [x_{k-1}, x_k]$ and $|x - y| < \delta$ we get

$$U(f, P) - L(f, P) = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) < \frac{\epsilon}{b - a} \sum_{k=1}^{n} (x_k - x_{k-1}) = \epsilon.$$

By the Cauchy criterion for integral we show that f is integrable. This finishes the proof.

3. Consider

$$f(x) = \int_0^{x^2} e^{\sqrt{t}} dt \quad \text{for } x \in [0, \infty).$$

(a) (4 points) Compute $f(0)$.

Solution:

$$f(0) = \int_0^0 e^{\sqrt{t}} dt = 0$$

since the interval of integration has length zero.

(b) (8 points) Show that f is differentiable on $(0, +\infty)$ and compute $f'(x)$.

Solution: The integrand $e^{\sqrt{t}}$ is a continuous function on $[0, \infty)$ and by the Fundamental theorem of Calculus II we have

$$F(x) = \int_0^x e^{\sqrt{t}} dt$$

is differentiable everywhere with $F'(x) = e^{\sqrt{x}}$. Now since $f(x) = F \circ g(x)$ where $g(x) = x^2$. By the chain rule we get

$$f'(x) = F'(g(x))g'(x) = e^{\sqrt{x^2}} 2x = 2xe^x.$$

4. Define $f : [0,2] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 3150 & \text{when } x \neq 1, \\ 0 & \text{when } x = 1. \end{cases}$$

(a) (4 points) Compute the lower Riemann sum $L(f)$.

Solution: Let P be a partition of $[0,2]$, say $P = \{0 = x_1 < x_2 < \ldots < x_n = 2\}$. Suppose for some l, $1 \leq l \leq n$, that $1 \in [x_{l-1}, x_l]$ and let $m_k = \inf_{[x_{k-1},x_k]} f$. Then

$$L(f,P) = \sum_{k=1}^{n} m_k(x_{k-1} - x_k)$$
$$= 3150(x_2 - x_1) + 3150(x_3 - x_2) + \ldots + 3150(x_{l-1} - x_{l-2}) + 0(x_l - x_{l-1}) + 3150((x_{l+1} - x_l) + \ldots + 3$$
$$= 3150(2-0) - 3150(x_l - x_{l-1}) = 3150(2 - (x_l - x_{l-1})).$$

Since

$$L(f) = \sup_P L(f,P)$$

and we can make $(x_l - x_{l-1})$ as small as we want by choosing the length of each subinterval small we see that

$$L(f) = \sup_P L(f,P) = 3150 * 2.$$

(b) (4 points) Compute the upper Riemann sum $U(f)$.

Solution: Let P be a partition of $[0,2]$, say $P = \{0 = x_1 < x_2 < \ldots < x_n = 2\}$. Suppose for some l, $1 \leq l \leq n$, that $1 \in [x_{l-1}, x_l]$ and let $M_k = \sup_{[x_{k-1},x_k]} f$. Then

$$L(f,P) = \sum_{k=1}^{n} m_k(x_{k-1} - x_k)$$
$$= 3150(x_2 - x_1) + 3150(x_3 - x_2) + \ldots + 3150(x_{l-1} - x_{l-2}) + 3150(x_l - x_{l-1}) + 2((x_{l+1} - x_l) + \ldots + 3$$
$$= 3150(x_n - x_1) = 3150 * 2.$$

(c) (7 points) Show that f is integrable on $[0,2]$ and find $\int_0^2 f(x)dx$.

Solution: Since

$$L(f) = inf_P L(f,P) = 3150 * 2 = \int_0^2 f(x)dx.$$

5. (12 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq C|x - y|^2 \quad \text{for all } x, y \in \mathbb{R}.$$

for some $C > 0$. Show that f is constant. (Hint: Compute first derivative of f).

Solution: Using the given assumption we get

$$\frac{|f(x) - f(y)|}{|x - y|} \leq C|x - y| \quad \text{for all } x, y \in \mathbb{R}.$$

Now if we take the limit as $x \to y$ we get

$$f'(y) = \lim_{x \to y} \frac{|f(x) - f(y)|}{|x - y|} \leq C \lim_{x \to y} |x - y| = 0$$

and therefore f is differentiable at every $y \in \mathbb{R}$ and $f'(y) = 0$. In class we showed that if $f'(y) = 0$ for all $y \in \mathbb{R}$ then $f =$ constant. Hence f is constant.

6. (12 points) Let $f : [0, +\infty) \to \mathbb{R}$ be a continuous and differentiable function satisfying

$$f(x) + xf'(x) \geq 0 \quad \text{for all } x > 0.$$

Show that $f(x) \geq 0$ for all $x \geq 0$. (Consider a function $g(x) = xf(x)$).

Solution: As the hint tells us we define $g(x) = xf(x)$ and then (since f is differentiable and x itself is also differentiable)

$$g'(x) = f(x) + xf'(x) \geq 0 \quad \text{for all } x > 0$$

by the given assumption. Therefore g is an increasing function on $(0, \infty)$. Also, $g(0) = 0f(0) = 0$ (since f is continuous it is continuous at zero). Hence we conclude that $g(x) \geq 0$ for $x \geq 0$. Now for $x > 0$ we consider $f(x) = g(x)/x$. Since $g(x) \geq 0$ and $x > 0$ then the ratio $g(x)/x$ is always positive for $x > 0$. For $x = 0$ we look the limit

$$\lim_{x \to 0} f(x) \geq 0.$$

This is because f is continuous function and non-negative for every $x > 0$ and hence limit of f will be non-negative.

7. (a) (8 points (bonus)) Let f and g be a continuous functions on $[a, b]$ such that $\int_a^b f = \int_a^b g$. Prove that there exists $x \in [a, b]$ such that $f(x) = g(x)$.

Solution: Let f and g be continuous functions on $[a, b]$ with $\int_a^b f = \int_a^b g$. Let $h(x) = f(x) - g(x)$. Then h is a continuous function on $[a, b]$ and $\int_a^b h = 0$. Let

$$M = \max_{[a,b]} h(x_1) \quad m = \min_{[a,b]} f = h(x_2).$$

Let P be the following partitioning of $[a, b]$, $P = \{a = x_0 < x_2 = b\}$. Then

$$0 = \int_a^b h = \leq U(h, P) = M(b - a) = h(x_1)(b - a)$$

and

$$0 = \int_a^b h = \geq L(h, P) = m(b - a) = h(x_2)(b - a)$$

There are three cases to consider. If $0 = h(x_1)(b - a)$ we get $(x_1) = 0$ which implies $f(x_1) = g(x_1)$ we are done. If this is not the case then $h(x_1) > 0$. Now if $0 = h(x_2)(b - a)$ then $h(x_2) = 0$ and this implies $f(x_2) = g(x_2)$ and we are done. If this is not the case then $h(x_2) < 0$. Now by the intermediate value theorem there exists x between x_1 and x_2 such that $h(x) = 0$ which implies $g(x) = f(x)$. Hence in all cases we prove that that there exists $x \in [a, b]$ such that $f(x) = g(x)$.

(b) (8 points (bonus)) Construct an example of integrable functions f and g on $[a, b]$ where $\int_a^b f = \int_a^b g$ but $f(x) \neq g(x)$ for all $x \in [a, b]$.

Solution: Note that continuity in the first part crucial. So f and g in this part can not be continuous. On the interval $[-1, 1]$ consider

$$f(x) = \begin{cases} -1 & \text{when } -1 \leq x < 0, \\ 1 & \text{when } 0 \leq x \leq 1 \end{cases}$$

and

$$f(x) = \begin{cases} 1 & \text{when } -1 \leq x < 0, \\ -1 & \text{when } 0 \leq x \leq 1 \end{cases}$$

Then $f(x) \neq g(x)$ for any $x \in [-1, 1]$. On the other hand you can compute that $U(f) = L(f) = 0 = U(g) = L(g)$.