UCONN

UNIVERSITY OF CONNECTICUT

Fall 2018 - Math 3150 **Name (Print):** ________________________

Practice Exam 1 - September 18

Time Limit: 75 Minutes

This exam contains 7 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	9	
2	10	
3	10	
4	9	
5	10	
6	7	
7	10	
8	10	
Total:	75	

1. For each of the following statements, say whether it is true or false. If the statement is false, give a counterexample.

   (a) (3 points) Convergent sequences are bounded.

   Solution: This is a true statement. This is Theorem 9.2.

   (b) (3 points) Every bounded sequence is convergent.

   Solution: This is a false statement. For example, consider the sequence (a_n) given by $a_n = (-1)^n$. We showed in class that it is not a convergent sequence yet it is bounded by above 1 and below -1.

   (c) (3 points) The limit of a convergent sequence of negative numbers is negative.

   Solution: This is also a false statement. Consider the sequence (a_n) given by $a_n = -\frac{1}{n}$. We showed that $\lim a_n = 0$ yet $a_n < 0$ for every $n \in \mathbb{N}$.

2. (10 points) Using mathematical induction prove that $n^2 + 5n + 1$ is an odd integer for every positive integer n.

Solution: Let P_n be the statement "$n^2 + 5n + 1$ is an odd integer.". We first observe the base step, $n = 1$.

P_1: when $n = 1$ we have $1^2 + 5 + 1 = 7$ which is odd. Therefore P_1 is true.

Assume that P_n is also true;"$n^2 + 5n + 1$ is an odd integer" then we consider P_{n+1};

$$(n+1)^2 + 5(n+1) + 1 = n^2 + 2n + 1 + 5n + 5 + 1 = n^2 + 5n + 1 + 2n + 6$$

Notice that by P_n we know that $n^2 + 5n + 1$ is odd. On the other hand, $2n + 6 = 2(n+3)$ is obviously even. We conclude that odd + even is odd. Hence P_{n+1} is true. By mathematical induction we conclude that our statement $n^2 + 5n + 1$ is an odd integer for every positive integer n is a true statement.

3. (10 points) Prove that $1 + \sqrt{1 + \sqrt{2}}$ is irrational.

Solution: Suppose $x = 1 + \sqrt{1 + \sqrt{2}}$. Then $x - 1 = \sqrt{1 + \sqrt{2}}$ or $(x-1)^2 = 1 + \sqrt{2}$ or $(x-1)^2 - 1 = \sqrt{2}$ or equivalently $((x-1)^2 - 1)^2 = 2$. From this we have that x is a solution to

$$(x^2 - 2x + 1 - 1)^2 - 2 = x^4 - 4x^3 + 4x^2 - 2 = 0.$$

From the corollary of the Rational zeros theorem we know that any solution of this equation must be an integer which divides 2. The only possible rational solutions are ± 1 and ± 2. It is clear that none of these numbers are solutions. Therefore x is not a rational number.

4. Let $S = \{x \text{ irrational};\ 1 \leq x \leq 3\}$.
   (a) (3 points) What are $\sup(S)$ and $\inf(S)$?

   Solution: $\sup(S) = 3$ and $\inf(S) = 1$.

   (b) (3 points) Does the set S have a maximum element and/or a minimum element?

   Solution: The set does not have a maximum element as $\sup(S) = 3$ and $3 \notin S$. With the same reasoning, since $\inf(S) = 1$ and $1 \notin S$ the set does not have a minimum element.

   (c) (3 points) Write an increasing sequence $s_n \in S$, i.e. $s_n \leq s_{n+1}$ for every $n \in \mathbb{N}$ for which s_n converges to $\sup(S)$. (You found $\sup(S)$ in part (a)].

   Solution: We know that $\sqrt{2}$ is not a rational number and therefore if we consider $s_n = 3 - \sqrt{2}/n$ for $n \in \mathbb{N}$ we see that $s_n \to 3 = \sup(S)$ with each $s_n \in S$ for every $n \in \mathbb{N}$.

5. (10 points) Consider the sequence (s_n) defined for $n \in \mathbb{N}$ by

$$s_n = \begin{cases} 1 & \text{when } n \text{ is odd,} \\ 2 & \text{when } n \text{ is even.} \end{cases}$$

Show that (s_n) does not converge.

Solution: We prove this by contradiction. Suppose that $\lim s_n = s$ and let $\epsilon = 1$. Then using the definition of convergence we can find N large enough so that $|s_n - s| < \epsilon/2 = 1/2$. When $n > N$ and even we have $|s_n - s| = |2 - s| < 1/2 = \epsilon/2$ and when $n > N$ is odd we have $|s_n - s| = |1 - s| < 1/2 = \epsilon/2$. Using these observations and triangle inequality we have

$$1 = |2 - 1| = |2 - s + s - 1| \leq |2 - s| + |s - 1| < 1/2 + 1/2 = 1.$$

We got a false statement $1 < 1$. Therefore our assumption that $\lim s_n = s$ is false. Hence (s_n) does not converge.

6. (7 points) Suppose that $\lim a_n = a$ and $\lim b_n = b$ for some real numbers a and b.

$$\text{Find } \lim \frac{a_n^{3150} + b_n^{2018}}{a_n^2 + b_n^2 + 1}.$$

Justify all steps.

Solution: We know that $\lim a_n$ and $\lim b_n$ exist then we have

$$\lim \frac{a_n^{3150} + b_n^{2018}}{a_n^2 + b_n^2 + 1} = \frac{\lim a_n^{3150} + \lim b_n^{2018}}{\lim a_n^2 + \lim b_n^2} = \frac{a^{3150} + b^{2018}}{a^2 + b^2 + 1}.$$

7. (10 points) Using ϵ-definition, show that

$$\lim \frac{n-5}{n^2+7} = 0.$$

Solution: Given $\epsilon > 0$ we want to find integer N large enough so that

$$\left|\frac{n-5}{n^2+7}\right| < \epsilon.$$

Let us focus on $\frac{n-5}{n^2+7}$ a little. First observe that $n-5 < n$ for every $n \in \mathbb{N}$ and $n^2+7 \geq n^2$ for every $n \in \mathbb{N}$. From these observations we have

$$\left|\frac{n-5}{n^2+7}\right| \leq \frac{n}{n^2} = \frac{1}{n} < \epsilon.$$

Or equivalently, $1/\epsilon < n$. Hence given $\epsilon > 0$, choose $N = 1/\epsilon$ and if $n > N = 1/\epsilon$ we have

$$\epsilon > \frac{1}{n} = \frac{n}{n^2} \geq \left|\frac{n-5}{n^2+7}\right| = \left|\frac{n-5}{n^2+7} - 0\right|.$$

This shows that

$$\lim \frac{n-5}{n^2+7} = 0.$$

8. (10 points) Using the definition 9.8, (given $M > 0$, there is a number N such that $n > N$ implies $s_n > M$) show that

$$\lim \frac{n^2+3}{n+1} = \infty.$$

Solution: We are asked to use the definition to prove this. Therefore, given $M > 0$ we will find N in terms of N so that if $n > N$ one has

$$M < \frac{n^2+3}{n+1}.$$

Focus on $\frac{n^2+3}{n+1}$. Since $n^2 + 3 \geq n^2$ for all $n \in \mathbb{N}$. Also $n + 1 \leq n + n = 2n$. Hence

$$\frac{n^2+3}{n+1} \geq \frac{n^2}{2n} = \frac{n}{2} > M$$

From this we observe that it is enough to choose $N = 2M$. In this case if $n > N = 2M$ then $n/2 > M$ and therefore we have

$$M < \frac{n}{2} = \frac{n^2}{2n} \leq \frac{n^2+3}{n+1}.$$

This shows that

$$\lim \frac{n^2+3}{n+1} = \infty.$$