Fall 2018 - Math 3150
Practice Exam 2 - October 30
Time Limit: 75 Minutes

Name (Print): ____________________

This exam contains 8 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	16	
2	13	
3	12	
4	12	
5	10	
6	12	
Total:	75	

1. Let $s_1 = 1$ and $s_{n+1} = [1 - \frac{1}{(n+1)^2}]s_n$ for $n \geq 1$.

   (a) (4 points) Find s_2, s_3, and s_4.

   Solution: $s_2 = 3/4$, $s_3 = 2/3$, $s_4 = 5/8$.

   (b) (8 points) Use induction to show $s_n = \frac{n+1}{2n}$ for all $n \in \mathbb{N}$.

   Solution: We start with the base case, $n = 1$, then $s_1 = \frac{1+1}{2} = 1$. Hence base case is true. Assume that $s_n = \frac{n+1}{2n}$ and consider s_{n+1}.

$$\begin{aligned} s_{n+1} &= [1 - \frac{1}{(n+1)^2}]s_n = [1 - \frac{1}{(n+1)^2}]\frac{n+1}{2n} \\ &= [\frac{(n+1)^2 - 1}{(n+1)^2}\frac{n+1}{2n} = \frac{((n+1)-1)((n+1)+1)}{2n(n+1)} = \frac{n(n+2)}{2n(n+1)} = \frac{n+2}{2(n+1)}. \end{aligned}$$

   This shows that our statement is also true for $n + 1$. By mathematical induction $s_n = \frac{n+1}{2n}$ for every $n \in \mathbb{N}$.

   (c) (4 points) Find $\lim s_n$.

   Solution: Since we just showed that $s_n = \frac{n+1}{2n}$ then it is clear that

$$\lim s_n = \lim \frac{n+1}{2n} = \frac{1}{2}.$$

2. (13 points) Let (a_n) and (b_n) be two convergent sequences of real numbers both converging to the same real number $r \in \mathbb{R}$. Let (c_n) be a sequence defined as $c_{2n-1} = a_n$ and $c_{2n} = b_n$ for all $n \in \mathbb{N}$, i.e., $(c_n) = (a_1, b_1, a_2, b_2, \ldots,)$. Then show that $(c_n) \to r$.

Solution: We first observe that $c_{2n}a_n$ converges to r and $c_{2n-1} = b_n$ converges to r as well. Let $\epsilon > 0$ be given.

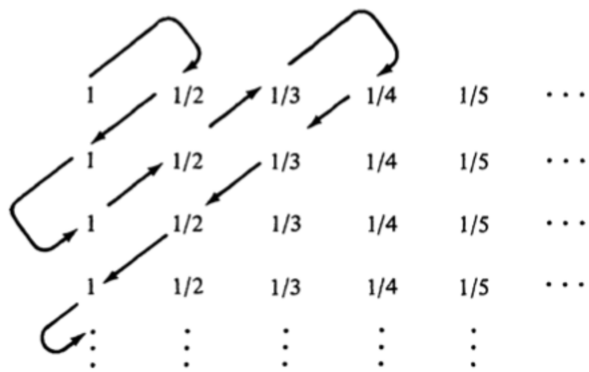
For this $\epsilon > 0$ (since c_{2n} converges to r then we can find N_1 for which if $n > N_1$ then $|c_{2n} - r| = |a_n - r| < \epsilon$. Similarly, for the same ϵ we can find N_2 such that $2n > N_2$ then $|c_{2n-1} - r| = |b_n - r| < \epsilon$.

Now let $N = 2\max\{N_1, N_2\}$ and if $n > N$ then

$$|c_n - r| < \epsilon.$$

This shows that (c_n) converges to r.

3. Let (s_n) be the sequence given in the figure.



(a) (6 points) Find the set S of the subsequential limits of (s_n).

Solution: It is clear from the picture that we can find subsequences converging to $1/n$ for every $n \in N$ and also the sequence itself is converging to 0. Therefore,

$$S = \{\frac{1}{n};\ n \in \mathbb{N}\} \cup \{0\}.$$

(b) (6 points) Determine $\limsup s_n$ and $\liminf s_n$.

Solution: By definition of $\limsup$ and $\liminf$ we know that $\limsup$ is the largest element in the set S. Hence $\limsup s_n = 1$ and with similar reasoning $\liminf s_n = 0$.

4. (a) (6 points) Find a decreasing sequence of (non empty) closed sets $F_1 \supset F_2 \supset \cdots \supset F_n \supset F_{n+1} \supset \ldots$ such that
$$\cap_{n=1}^{\infty} F_n = \emptyset.$$

Solution: Consider the following closed sets;

$$\begin{aligned} F_1 &= [1, \infty), \\ F_2 &= [2, \infty), \\ &\vdots \\ F_n &= [n, \infty), \\ &\vdots. \end{aligned}$$

Now the intervals are nested and closed as for example $F_1 \supset F_2$ and

$$F_1 \supset F_2 \supset \cdots \supset F_k \supset F_{k+1} \supset \ldots.$$

But their intersection is empty set as

$$\cap_{n=1}^{\infty} F_n = \cap_{n=1}^{\infty} [n, \infty) = \emptyset$$

as n diverges to ∞.

(b) (6 points) Find a decreasing sequence of (non empty) open bounded intervals $I_1 \supset I_2 \supset \cdots \supset I_n \supset I_{n+1} \supset \ldots$ such that
$$\cap_{n=1}^{\infty} I_n = \emptyset.$$

Solution: Consider the set

$$\begin{aligned} I_1 &= (0, 1), \\ I_2 &= (0, \frac{1}{2}), \\ I_3 &= (0, \frac{1}{3}), \\ &\vdots, \\ I_n &= (0, \frac{1}{n}), \\ &\vdots. \end{aligned}$$

The intervals are nested and bounded but

$$\cap_{n=1}^{\infty} I_n = \cap_{n=1}^{\infty} (0, \frac{1}{n}) = \emptyset$$

as $1/n$ converges to 0.

5. (10 points) Let (s_n) be any sequence. Show that there exists a monotonic subsequence whose limit is $\limsup s_n$.

Solution: See Theorem 11.7 from the lecture notes or from the book.

6. (12 points) The Dirichlet function $f : \mathbb{R} \mapsto \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{when } x \in \mathbb{Q}, \\ 0 & \text{when } x \notin \mathbb{Q}. \end{cases}$$

Show that f is discontinuous at every $x \in \mathbb{R}$.

Solution: Let $x \in R$ be fixed and $\epsilon = 1/2$ and $\delta > 0$. Then we will show that there exists $x_0 \in (x - \delta, x + \delta)$ (i.e., $|x - x_0| < \delta$) but

$$|f(x) - f(x_0)| = 1 > 1/2$$

which gives us desired conclusion.

If x is a rational then for every $\delta > 0$ we can always find a irrational (by density of irrationals) x_0 in $(x - \delta, x + \delta)$. Now

$$|f(x) - f(x_0) = |1 - 0| = 1 > 1/2$$

If x is irrational then with the same reasoning for every $\delta > 0$ we can always find a rational (by density of rationals) x_0 in $(x - \delta, x + \delta)$. Now

$$|f(x) - f(x_0) = |0 - 1| = 1 > 1/2$$