UCONN
UNIVERSITY OF CONNECTICUT

Fall 2018 - Math 3150
Practice Exam 2 - October 30
Time Limit: 75 Minutes

Name (Print): ______________________________

This exam contains 8 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	12	
2	12	
3	12	
4	15	
5	12	
6	12	
Total:	75	

Do not write in the table to the right.

1. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^3 \sin(\frac{1}{x^2}) & \text{when } x \neq 0, \\ 0 & \text{when } x = 0. \end{cases}$$

(a) (6 points) Find $f'(x)$ for $x \in \mathbb{R} \setminus \{0\}$.

Solution: We first observe that when $x \neq 0$ then $1/x^2$ is differentiable and $\sin(x)$ is differentiable everywhere. Now $\sin(1/x^2)$ is a composition of $\sin(x)$ and $1/x^2$ and therefore $x^3 \sin(\frac{1}{x^2})$ is differentiable at every $x \neq 0$ and

$$f'(x) = 3x^2 \sin(\frac{1}{x^2}) - 2\sin(\frac{1}{x^2}) \quad \text{whenever } \ x \in \mathbb{R} \setminus \{0\}.$$

(b) (6 points) Show that f is differentiable at $x = 0$ and find $f'(0)$.

Solution: To show that f is differentiable at 0 we look at the following limit

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^3 \sin(\frac{1}{x^2}) - 0}{x - 0} = \lim_{x\to 0} \frac{x^3 \sin(\frac{1}{x^2})}{x} \\ &= \lim_{x\to 0} x^2 \sin(\frac{1}{x^2}). \end{aligned}$$

One can show that

$$|x^2 \sin(\frac{1}{x^2})| \leq |x^2|$$

and as $x \to 0$ $x^2 \to 0$. Hence

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} x^2 \sin(\frac{1}{x^2}) = 0.$$

(c) (5 points (bonus)) Show that f' is not continuous at $x = 0$.

Solution: Now we found $f'(x)$ for $x \in \mathbb{R} \setminus \{0\}$ in (a) and $f'(0)$ in (b) and combining them we have

$$f'(x) = \begin{cases} 3x^2 \sin(\frac{1}{x^2}) - 2\sin(\frac{1}{x^2})) & \text{when } x \neq 0, \\ 0 & \text{when } x = 0. \end{cases}$$

We need to check if

$$\lim_{x\to 0} f'(x) = f'(0) = 0$$

to check continuity of f' at 0. Now

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} 3x^2 \sin(\frac{1}{x^2}) - 2\sin(\frac{1}{x^2}))$$

this limit does not exits because of $2\sin(\frac{1}{x^2}))$ does not have a limit as $x \to 0$. Hence f' is not continuous at $x = 0$.

2. (12 points) Prove that every monotone function on $[a, b]$ is integrable on $[a, b]$.

Solution: Let f be a monotone function on $[a, b]$. We prove the theorem when f is increasing and the decreasing one will be similar. We also assume that f is not constant on $[a, b]$ (i.e. $f(a) < f(b)$) otherwise we know that constant function is integrable. Given $\epsilon > 0$ let P be a partitioning of $[a, b]$, $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ such that

$$x_k - x_{k-1} < \frac{\epsilon}{f(b) - f(a)}.$$

Notice that since f is increasing we have

$$M_k = \sup_{[x_{k-1}, x_k]} f = f(x_k) \quad \text{and} \quad m_k = \inf_{[x_{k-1}, x_k]} f = f(x_{k-1}).$$

Then

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{k=1}^{n} M_k(x_k - x_{k-1}) - m_k(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}) - f(x_{k-1})(x_k - x_{k-1}) \\ &\le \frac{\epsilon}{f(b) - f(a)} \sum_{k=1}^{n} f(x_k) - f(x_{k-1}) \\ &= \frac{\epsilon}{f(b) - f(a)} (f(b) - f(a)) = \epsilon. \end{aligned}$$

Hence we prove that given $\epsilon > 0$ there exists a partitioning P one has

$$U(f, P) - L(f, P) < \epsilon.$$

This shows that f is integrable.

3. Let $f : [-\pi, \pi] \to \mathbb{R}$ be the function defined by

$$f(x) = \int_0^{x^2} e^{\sin(t)} dt.$$

(a) (4 points) Compute $f(0)$.

Solution: $f(0) = \int_0^0 e^{\sin(t)} dt = 0$ as the length of the interval of integration is zero.

(b) (8 points) Show that f is differentiable and compute $f'(x)$.

Solution: We first consider

$$F(y) = \int_0^{y} e^{\sin(t)} dt.$$

Since $e^{\sin(t)}$ is a continuous function everywhere by the Fundamental Theorem of Calculus II we get

$$F'(y) = e^{\sin(y)}.$$

Since $f(x) = F \circ g(x)$ where $g(x) = x^2$ and using chain rule we get

$$f'(x) = F'(g(x))g'(x) = e^{\sin(x^2)} 2x.$$

4. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 2 & \text{when } x \neq \frac{1}{2}, \\ 0 & \text{when } x = \frac{1}{2}. \end{cases}$$

(a) (4 points) Compute the lower Riemann sum $L(f)$.

Solution: Let P be a partition of $[0,1]$, say $P = \{0 = x_1 < x_2 < \ldots < x_n = 1\}$. Suppose for some l, $1 \leq l \leq n$, that $1/2 \in [x_{l-1}, x_l]$ and let $m_k = \inf_{[x_{k-1},x_k]} f$. Then

$$L(f,P) = \sum_{k=1}^{n} m_k(x_{k-1} - x_k)$$
$$= 2(x_2 - x_1) + 2(x_3 - x_2) + \ldots + 2(x_{l-1} - x_{l-2}) + 0(x_l - x_{l-1}) + 2((x_{l+1} - x_l) + \ldots + 2(x_n - x_{n-1})$$
$$= 2(1-0) - 2(x_l - x_{l-1}) = 2(1 - (x_l - x_{l-1})).$$

Since

$$L(f) = \sup_P L(f,P)$$

and we can make $(x_l - x_{l-1})$ as small as we want by choosing the length of each subinterval small we see that

$$L(f) = \sup_P L(f,P) = 2.$$

(b) (4 points) Compute the upper Riemann sum $U(f)$.

Solution: Let P be a partition of $[0,1]$, say $P = \{0 = x_1 < x_2 < \ldots < x_n = 1\}$. Suppose for some l, $1 \leq l \leq n$, that $1/2 \in [x_{l-1}, x_l]$ and let $M_k = \sup_{[x_{k-1},x_k]} f$. Then

$$L(f,P) = \sum_{k=1}^{n} m_k(x_{k-1} - x_k)$$
$$= 2(x_2 - x_1) + 2(x_3 - x_2) + \ldots + 2(x_{l-1} - x_{l-2}) + 2(x_l - x_{l-1}) + 2((x_{l+1} - x_l) + \ldots + 2(x_n - x_{n-1})$$
$$= 2.$$

Since

$$L(f) = inf_P L(f,P) = 2$$

(c) (7 points) Show that f is integrable on $[0,1]$ and find $\int_0^1 f(x)dx$.

Solution: We just showed in (a) and (b) that

$$L(f) = U(f) = 2.$$

Hence f is integrable on $[0,1]$ and

$$\int_0^1 f = 2.$$

5. (a) (6 points) Show that for all $x, y \in \mathbb{R}$

$$|\sin(x) - \sin(y)| \leq |x - y|.$$

Solution: Let $f(x) = \sin(x)$ for $x, y \in \mathbb{R}$. Consider

$$\frac{f(x) - f(y)}{x - y}.$$

By the Mean Value theorem there exists c such that

$$\frac{f(x) - f(y)}{x - y} = f'(c) = \cos(c).$$

Then

$$\left|\frac{f(x) - f(y)}{x - y}\right| = |\cos(c)| \leq 1.$$

This gives

$$|f(x) - f(y)| \leq |x - y| \quad \text{which is} \quad |\sin(x) - \sin(y)| \leq |x - y|.$$

(b) (6 points) Is the function $f(x) = \sin(x)$ uniformly continuous on $\mathbb{R}$?

Solution: It is uniformly continuous on $\mathbb{R}$ since $f'(x) = \cos(x)$ and $|f'(x)| \leq 1$ for all x. We can prove uniform continuity by using the definition. Given $\epsilon > 0$ choose $\delta = \epsilon$ and using part (a) we have

$$|\sin(x) - \sin(y)| \leq |x - y| < \delta = \epsilon$$

whenever $|x - y| < \delta$. Another way to prove this is to remember that $\sin(x)$ is periodic function with period 2π. It is enough to show uniform continuity in $[0, 2\pi]$ which we already know from (a) and hence $\sin(x)$ is uniformly continuous on $\mathbb{R}$.

6. (12 points) Prove that $1 + x < e^x$ for all $x > 0$.

Solution: Let $f(x) = e^x - x - 1$. First check that $f(0) = e^0 - 0 - 1 = 0$. Now if we can show that f is strictly increasing function then $0 = f(0) < f(x)$ for $x > 0$ then this implies that

$$0 < f(x) = e^x - x - 1 \quad \text{or} \quad 1 + x < e^x.$$

Hence we need to show that f is strictly increasing. For this we look if $f'(x) > 0$.

$$f'(x) = e^x - 1$$

which is positive for every $x > 0$. Hence $f'(x) > 0$ for $x > 0$ and this finishes the proof.