Second Order Linear Equations

In this part of the lecture we study second order linear DE:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$

where $a_2(x) \not\equiv 0$ with some initial or boundary conditions.

If $b(t) \equiv 0$ then the DE is called homogeneous otherwise it's called non-homogen

Homogeneous Equations with Constant Coefficients

The DE we consider first is

$$a_2 y'' + a_1 y' + a_0 y = b$$

here a_2, a_1, a_0 and b are constan

In this case we know that the characteristic equation is

$$a_2 r^2 + a_1 r + a_0 = 0$$

and suppose r_1 & r_2 are solutions to this characteristic equation we have the general solutions

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$ → Rember this from last year!

Example: Consider the DE

$y'' - 5y' + 6y = 0$ with initial conditions $y(0) =$

what happens as $x \to \infty$ or $x \to -\infty$

Solution: Since characteristic equation is $r^2 - 5r + 6 = 0$

we get $(r-3)(r-2) = 0$

Hence $r_1 = 2$ & $r_2 = 3$

therefore $y(x) = c_1 e^{2x} + c_2 e^{3x}$ is the general solution

$y(0) = c_1 + c_2 = -1$

$y'(0) = 2c_1 + 3c_2 = 1$

solve c_1, c_2 $\Rightarrow$ $c_1 = -4$, $c_2 = 3$

Hence particular solution is $y(x) = -4e^{2x} + 3e^{3x}$.

Since e^{2x} & $e^{3x} \to \infty$ one gets (e^{3x} dominates e^{2x})

$y(x) \to \infty$ as $x \to \infty$

$y(x) \to 0$ as $x \to -\infty$

Solutions of Linear Homogeneous Equations
The Wronskian

Thm: [Existence & Uniqueness Thm]
Consider the following DE:

$$y'' + p(x)\, y' + q(x)\, y = g(x) \quad \text{with}$$
$$y(x_0) = y_0$$
$$y'(x_0) = \overline{y}_0$$

If $p(x)$, $q(x)$, $g(x)$ are continuous and bounded on an interval I containing t_0, then there is exactly one solution $y(t)$ to this equation valid on I.

Example: $(t^2 - 3t)\, y'' + t y' - (t+3)\, y = e^t$
$$y(1) = 2 \qquad y'(1) = 1$$

Find the largest interval including ~~the~~ 1 for which solution is valid.

Solution: Rewrite the DE
$$y'' \cancel{+} \cdot \frac{t}{t^2-3t}\, y' - \frac{(t+3)}{(t^2-3t)}\, y = \frac{e^t}{(t^2-3t)}$$

Hence $p(t) = \dfrac{*}{t^2-3t} = \dfrac{*}{t(t-3)}$

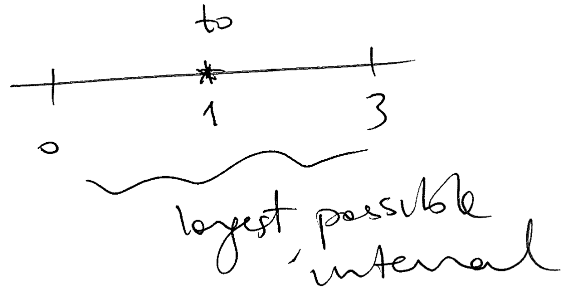
$q(t) = -\dfrac{t+3}{t^2-3t}$

$g(t) = \dfrac{e^t}{t(t-3)}$

Hence $t \neq 0$ and $t \neq 3$

As $t_0 = 1$

then the



Hence $I = (0,3)$

On $I = (0,3)$ by Thm $\exists!$ unique solution to above DE initial conditions

The Wronskian:

Given two functions say f, g the Wronskian is defined as $W(f,g)(x) = fg' - gf'$

$$= \det \begin{bmatrix} f & g \\ f' & g' \end{bmatrix}$$

Good properties of W:

*) if $W(f,g) = 0$ then f and g are linearly dependent

2) otherwise they are linearly independent.

Example! check that if the pair of functions

$$f(x) = (\cancel{*}+1) \quad \text{and} \quad g(x) = x$$

are linearly independent.

Solution! $f(x) = x+1 \rightarrow f'(x) = 1$

$g(x) = x \rightarrow g'(x) = 1$

$$W(f,g) = \det \begin{bmatrix} x+1 & x \\ 1 & 1 \end{bmatrix} = x+1-x = 1 \neq 0$$

Hence f, g are linearly independent.

Theorem [Abel's Theorem]

Let y_1 and y_2 be two linearly independent solutions to the DE

$$y'' + p(x)y' + q(x)y = 0 \quad \text{on an interval}$$

Then the Wronskian $W(y_1, y_2)$ on I is given by

$$W(y_1, y_2)(x) = c \cdot e^{-\int p(x)\,dx}.$$

for some c depending on y_1, y_2 but independent of x or I.

Example! Without solving the DE

$$x^2 y'' - x(x+2)\, y' + (x+2) y = 0$$

find the Wronskian. $W(y_1, y_2)$.

Solution! We first need to ~~w~~ rewrite the DE by dividing x^2

$$y'' - \left(\frac{x(x+2)}{x^2}\right) y' + \left(\frac{x+2}{x^2}\right) y = 0$$

Hence $p(x) = -\frac{x+2}{x}$. which is defined when $x \neq 0$

From Abel's theorem we have

$$W(y_1, y_2) = C e^{-\int p(x)\,dx} = C e^{-\int \left(-\frac{x+2}{x}\right) dx}$$

$$= C e^{\int \left(1 + \frac{2}{x}\right) dx} = C e^{x + 2\ln|x|} = C e^{x} \cdot e^{\ln x^2}$$

$$= C e^{x} \cdot x^2$$

Notice that the Wronskian is defined on either $(-\infty, 0)$ or $(0, \infty)$. This tells us that the solutions y_1 and y_2 (we do not what they are) are defined on either $(-\infty, 0)$ or $(0, \infty)$.

Example! Consider the following DE

$$x^2 y'' - x(x+2) y' + (x+2) y = 0 \quad (\text{for } x>$$

It's given that $y_1(x) = x$ is a solution.

Find the general solution.

Solution: suppose y_2 is the second solution and once we find it

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$ will be

the general solution.

From Abel's theorem, we know that the Wronskian of y_1 and y_2 are

$$W(y_1, y_2) = e^{-\int p(x)\,dx} \quad \text{where } p(x) = \frac{-x(x+2)}{x^2}$$

$$= c e^x \cdot x^2 \quad (\text{from the previous exercise}).$$

On the other hand, by definition of Wronski

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2$$

$$= x y_2' - y_2 = c e^x \cdot x^2$$

Now if we rewrite the DE we get

$$y_2' - \frac{1}{x} y_2 = c e^x \cdot x$$

This can be written as

$$\left(y_2 \cdot \ln x\right)' = ce^x \cdot x$$

Integrate both sides to get

$$y_2(x) \cdot \ln x = c\int e^x \cdot x \, dx$$

$$= ce^x \cdot x - ce^x + c_2$$

$$\text{Hence } y(x) = c_1 x + c\left(e^x x - e^x\right) + c_2 .$$

Therefore eventhough we may not be able to solve the DE we can use the Wronskian to find the second solution.

Higher Order Linear Equations

The general form of a linear equation of order n with $y(x)$ unknown is

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \dots + p_1(x)\, y' + p_0(x)\, y = g(t).$$

Here we find a general solution which will contain n-parameters. And we need n conditions to find the particular solution.

Existence & Uniqueness: If all the coefficients $p_{n-1}(x) \dots p_0(x), g(x)$ are all continuous and bounded on an interval I then there exists a unique solution to this DE.

If $p_{n-1}(x), \dots p_0(x)$ are constant and $g(x) = 0$ then we get a homogeneous DE with constant coefficients.

In this case, one can find the corresponding characteristic equation to find the general solutions.

Example: $y''' - 6y'' + 11y' - 6y = 0$

the characteristic equation is

$$(r^3 - 6r^2 + 11r - 6) = 0$$

the roots are $r_1 = 1$, $r_2 = 2$ $r_3 = 3$

Hence the general solution is

$$y(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$

Laplace Transform

Given a function $f(x)$, $x \geq 0$, we will use $\mathcal{L}\{f(x)\}$ to denote its Laplace transform, defined as

$$F(s) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x)\,dx$$

$$= \lim_{t \to \infty} \int_0^t e^{-sx} f(x)\,dx$$

<u>Example:</u> Let $f(x) = 1$ for $x \geq 0$.

then $$F(s) = \mathcal{L}\{f(x)\} = \lim_{t \to \infty} \int_0^t e^{-sx}\, 1\, dx$$

$$= \lim_{t \to \infty} -\frac{1}{s}\left[e^{-st} - 1\right]$$

$$= \frac{1}{s} \quad s > 0.$$

Example: let $f(x) = x^n$ for $n \geq 1$ integer. Find the Laplace transform of f.

Solution: $F(s) = \mathcal{L}\{f(x)\} = \lim_{t\to\infty} \int_0^t e^{-sx} x^n \, dx$

$$= \lim_{t\to\infty} \int_0^t e^{-sx} x^n \, dx$$

$$= \lim_{t\to\infty} \left[\frac{x^n e^{-sx}}{-s} \Big|_0^t - \int_0^t \frac{n x^{n-1} e^{-sx}}{-s} \cdot dx \right]$$

$$= 0 + \frac{n}{s} \int_0^\infty e^{-sx} x^{n-1} \, dx = \frac{n}{s} \mathcal{L}\{x^{n-1}\}$$

Hence we get a recursive formula

$$\mathcal{L}\{x^n\} = \frac{n}{s} \mathcal{L}\{x^{n-1}\}$$

$$= \frac{n}{s} \frac{(n-1)}{s} \mathcal{L}\{x^{n-2}\} \cdots$$

$$= \frac{n}{s} \frac{(n-1)}{s} \cdots \frac{2}{s} \cdot \frac{1}{s} \mathcal{L}(1)$$

$$= \frac{n!}{s^n} \cdot \frac{1}{s} = \frac{n!}{s^{n+1}}.$$

We can also find laplace transform of piecewise functions.

<u>Example:</u> Find the laplace transform of

$$f(x) = \begin{cases} 1 & 0 \le \cancel{x} < 2 \\ \cancel{x} - 2 & 2 \le t \end{cases}$$

then $\mathcal{L}\{f(x)\} = F(s)$

$$= \lim_{t \to \infty} \int_0^t e^{-sx} f(x)\, dx$$

$$= \int_0^\infty e^{-sx} f(x)\, dx$$

$$= \int_0^2 e^{-sx} f(x)\, dx + \int_2^\infty e^{-sx} f(x)\, dx$$

$$= \int_0^2 e^{-sx}\, 1\, dx + \int_2^\infty e^{-sx} (x-2)\, dx$$

Now

$$F(s) = -\frac{1}{s} e^{-sx} \Big|_0^2 + (x-2)\frac{e^{-sx}}{-s} \Big|_2^\infty - \int_2^\infty \frac{e^{-sx}}{-s} dx$$

$$= -\frac{1}{s}(e^{-2s}-1) + \frac{1}{s^2} e^{-2s}.$$

Properties of Laplace Transform:

1) <u>Linearity</u>: $\mathcal{L}\{c_1 f(x) + c_2 g(x)\}$
$= c_1 \mathcal{L}\{f(x)\} + c_2 \mathcal{L}\{g(x)\}$

2) <u>First Derivative</u>: $\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f$

3) <u>Second Derivative</u>: $\mathcal{L}\{f''(x)\} = s^2 \mathcal{L}\{f(x)\}$
$- s f(0) -$

4) Higher order derivatives:

$$\mathcal{L}\{f^{(n)}(x)\} = s^n \mathcal{L}\{f(x)\} - s^{n-1} f(0) - s^{n-2} f'(0)$$
$$\dots s f^{(n-2)}(0) - f^{(n-1)}(0).$$

5) $\mathcal{L}\{-x f(x)\} = F'(s)$ where $\mathcal{L}\{f(x)\} = F(s)$.

6) $\mathcal{L}\{e^{ax} f(x)\} = F(s-a)$ where

$\mathcal{L}\{f(x)\} = F(s)$

(1)–(6) are exercise!.

<u>Example:</u> $\mathcal{L}\{e^{ax} . x^n\} = ?$

To find this we first find

$$\mathcal{L}\{x^n\} = \frac{n!}{s^{n+1}}$$

Then using property (6) we get

$$\mathcal{L}\{e^{ax} x^n\} = \frac{n!}{(s-a)^{n+1}}.$$

<u>Example!</u> Find $\mathcal{L}\{e^{2x}(x^3+5x-2)\}$

First find $\mathcal{L}\{x^3+5x-2\}$

$$= \frac{3!}{s^4} + \frac{5}{s^2} - \frac{2}{s}$$

Then use the property (6) to get

$$\mathcal{L}\{e^{2x}(x^3+5x-2)\} \qquad (a=2)$$

$$=\frac{3!}{(s-2)^4}+\frac{5}{(s-2)^2}-\frac{2}{s-2}.$$

<u>Example:</u> Show that laplace transform of $\cos at$ is $\frac{s}{s^2+a^2}$ $s>0$ and

$\sin at$ is $\frac{a}{s^2+a^2}$ $s>0$.

<u>Proof:</u> Remember that Euler's formula says

$$e^{iat}=\cos at+i\sin at$$

where i is the ~~complex~~ such that $i=\sqrt{-1}$

Then $\mathcal{L}\{e^{iat}\}=\mathcal{L}\{\cos at+i\sin at\}$

$$=\mathcal{L}\{\cos at\}+i\,\mathcal{L}\{\sin at\}$$

We know that $\mathcal{L}\{e^{iat}\cdot 1\}=\frac{1}{s-ia}$

$$=\frac{(s+ia)}{(s-ia)(s+ia)}=\frac{s+ia}{s^2+a^2}=\frac{s}{s^2+a^2}+i\frac{a}{s^2+a^2}$$

Solutions of Initial Value Problems

Example: Using Laplace transform

find the particular solution with the given initial value;

$$y'' - y' - 2y = e^{2x}, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution: Apply the Laplace transform both sides;

$$\mathcal{L}\{y'' - y' - 2y\} = \mathcal{L}\{e^{2x} \cdot 1\}$$

use Linearity (1)

$$= \mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = \frac{1}{s-2}.$$

$$- s\mathcal{L}\{y\} - 2\mathcal{L}\{y\} = \frac{1}{s-2}$$

$$s^2\mathcal{L}\{y\} - s\underbrace{y(0)}_{=0} - y'(0)$$

$$s^2\mathcal{L}\{y\} - \underbrace{1}_{y'(0)} - s\mathcal{L}\{y\} - 2\mathcal{L}\{y\} = \frac{1}{s-2}$$

Solve for $\mathcal{L}\{y\}$ to get

$$\mathcal{L}\{y\} = \frac{s-1}{(s-2)(s^2-s-2)}$$

and use partial fraction to get

$$\mathcal{L}\{y\} = \frac{s-1}{(s-2)^2(s+1)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

Find A, B, C $\quad A = -\frac{2}{9}$, $B = \frac{2}{9}$, $C = \frac{1}{3}$

Hence $\quad \mathcal{L}\{y(x)\} = \frac{-\frac{2}{9}}{s+1} + \frac{\frac{2}{9}}{s-2} + \frac{\frac{1}{3}}{(s-2)^2}$

Then apply inverse Laplace transform to get

$$\mathcal{L}^{-1}\{\mathcal{L}\{y(x)\}\} = y(x)$$

$$= \mathcal{L}^{-1}\left\{ \frac{-\frac{2}{9}}{s+1} + \frac{\frac{2}{9}}{s-2} + \frac{\frac{1}{3}}{(s-2)^2} \right)$$

$$= -\frac{2}{9}e^{-x} + \frac{2}{9}e^{2x} + \frac{1}{3}xe^{2x}.$$

Example: Solve the initial value problem using Laplace transform.

$$y'' + 3y' + 2y = 6e^{t} \qquad y(0)=2$$
$$y'(0)=-1.$$

Solution: Take the Laplace transform of both sides

$$\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{6e^{t}\}$$

using linearity and other properties we get

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 6\mathcal{L}\{e^{t}\}$$

$$s^2\mathcal{L}\{y(x)\} - sy(0) - y'(0) + 3s\mathcal{L}\{y(x)\} - y(0)$$
$$+ 2\mathcal{L}\{y\} = 6 \cdot \frac{1}{s-1}$$

Solve for $\mathcal{L}\{y(x)\}$ to get

$$\mathcal{L}\{y(x)\} = \frac{2s-4}{s^2+3s+2} + \frac{6}{s-1}$$

Now apply the inverse transform but first do some arrangement;

$$\mathcal{L}\{y(x)\} = \frac{8}{s+2} - \frac{6}{s+1} + \frac{6}{s-1}$$

$$y(x) = \mathcal{L}^{-1}\left\{\frac{8}{s+2} - \frac{6}{s+1} + \frac{6}{s-1}\right\}$$

$$= 8\,\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - 6\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 6\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= 8e^{-2x} - 6e^{-x} + 6e^{x}$$ is the solution.

<u>Exercises</u>: Solve the following DE using Laplace transform

1) $y'' - 2y' + 2y = e^{-x}$ $\quad y(0) = 0$, $\quad y'(0) = 1$

<u>Answer</u>: $\frac{1}{5}e^{-x} - \frac{1}{5}e^{x}\cos x + \frac{9}{5}e^{x}\sin x = y(x)$

2) $y'' + y = \cos 2t$, $\quad y(0) = 2$, $\quad y'(0) = 1$

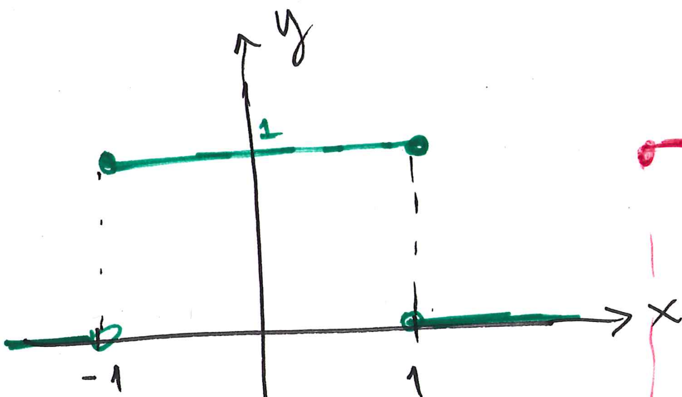
<u>Answer</u>: $y(x) = \frac{7}{3}\cos x + \sin x - \frac{1}{3}\cos 2x$.

More on Laplace Transform

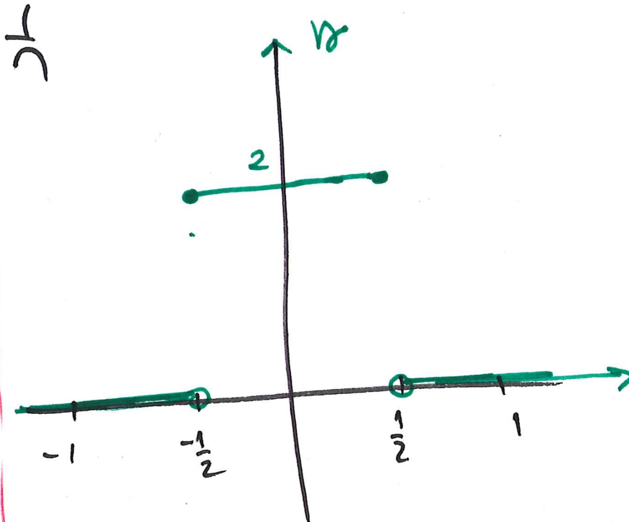
Dirac-Delta Function

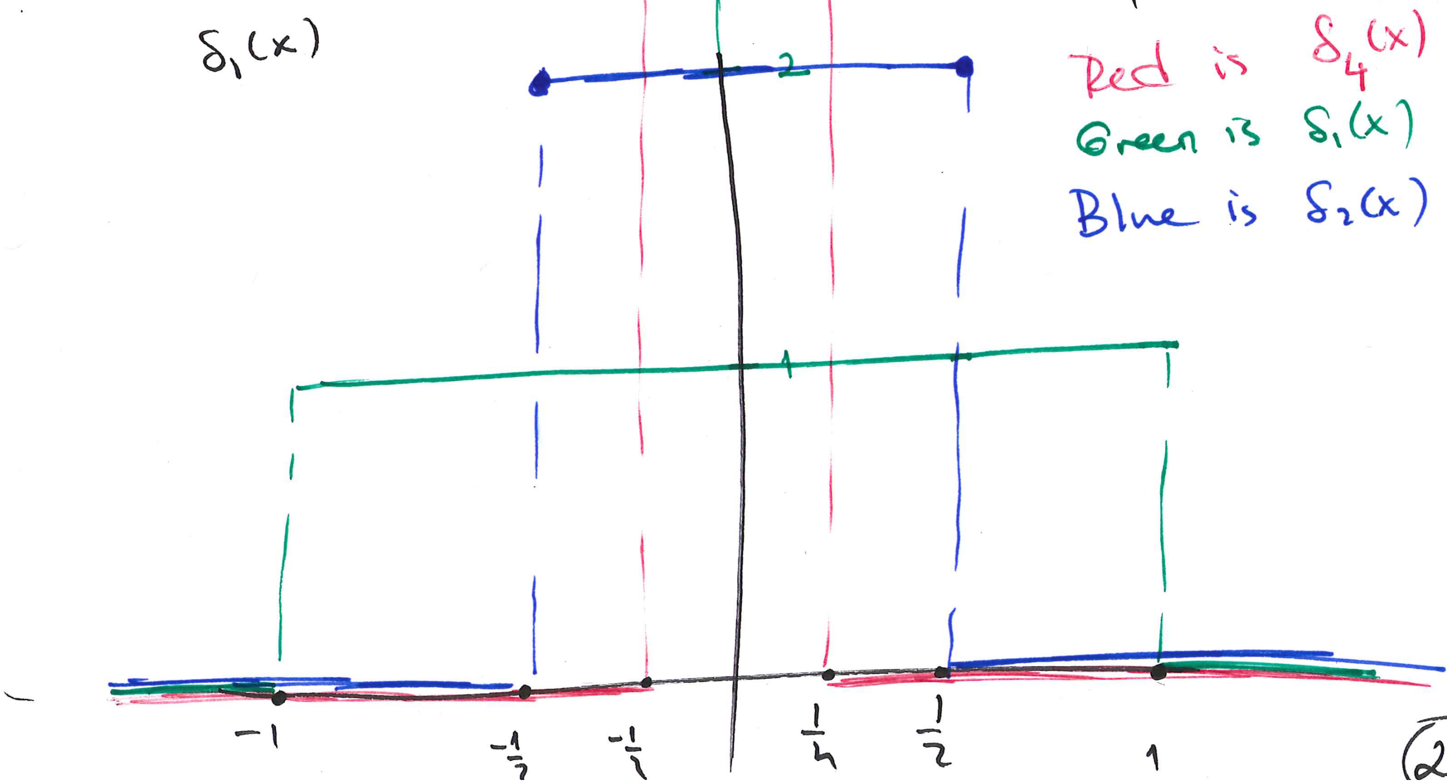
Let's define a sequence of functions

as
$$\delta_n(x) = \begin{cases} 0 & x < -\frac{1}{n} \\ n & -\frac{1}{n} \le x \le \frac{1}{n} \\ 0 & x > \frac{1}{n} \end{cases} \qquad \text{for } n = 1, 2 \ldots$$



$\delta_1(x)$





δ_n converges to a function called Dirac-Delta distribution

$$\delta(x) = \begin{cases} 0, & \cancel{x} \neq 0 \\ \infty, & \cancel{x} = 0 . \end{cases}$$

Some interesting properties

1) $\int_{-\infty}^{\infty} \delta(x)\,dx = 1 \rightarrow$ Area below δ.

2) $\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$ for any cont. functi f.

3) $\delta_{x_0}(x) = \delta(x - x_0) = \begin{cases} 0 & x \neq x_0 \\ \infty & x = x_0 \end{cases}$

$\int_{-\infty}^{\infty} f(x)\,\delta_{x_0}(x)\,dx = f(x_0).$

$$\mathcal{L}\{\delta_{x_0}(x)\} = \int_0^{\infty} e^{-sx}\,\delta_{x_0}(x)\,dx = e^{-sx_0}.$$

Example: solve the following DE

$$y'' + 4y = \delta_{\pi}(x) - \delta_{2\pi}(x) \qquad y(0) = y'(0) = 0$$

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{\delta_{\pi}(x) - \delta_{2\pi}(x)\}$$

$$s^2 \mathcal{L}\{y(x)\} - s y(0) - y'(0) + 4\mathcal{L}\{y(x)\}$$
$$= \mathcal{L}\{\delta_{\pi}(x)\} - \mathcal{L}\{\delta_{2\pi}(x)\}$$
$$= e^{-s\pi} - e^{-2\pi s}$$

$$s^2 \mathcal{L}\{y(x)\} + 4\mathcal{L}\{y(x)\} = e^{-s\pi} - e^{-2\pi s}$$

Hence $\mathcal{L}\{y(x)\} = \dfrac{e^{-s\pi}}{s^2+2^2} - \dfrac{e^{-2\pi s}}{s^2+2^2}$

At this point we can not really find

$$y(x) = \mathcal{L}^{-1}\left\{ \frac{e^{-s\pi}}{s^2+2^2} - \frac{e^{-2\pi s}}{s^2+2^2} \right\}$$

as we have product of two function depen[ding]
on s.

To solve this we need to find

$$\mathcal{L}^{-1}\{F(s)\,G(s)\}.$$

Suppose $\mathcal{L}\{f(x)\} = F(s)$

$\mathcal{L}\{g(x)\} = G(s).$

<u>Definition</u> [Convolution] let $f(x)$ and $g(x)$ be two functions, then $(f*g)(x)$ is called f convolution g and is

$$(f*g)(x) = \int_0^x f(x-t)\,g(t)\,dt$$

Properties of convolution operator $*$.

* Commutativity: $f*g = g*f$
* Distributivity $f*(g_1+g_2) = f*g_1 + f*$
* Associativity $f*(g*h) = (f*g)*h$.

Now

__Theorem:__ let $\mathcal{L}\{f(x)\} = F(s)$

$\mathcal{L}\{g(x)\} = G(s)$

then $$\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(x).$$

__Proof:__ Integrating by parts!

__Example:__ $f(x) = 3x$, $g(x) = \sin 5x$

$$(f*g)(x) = \int_0^x f(x-t)\,g(t)\,dt$$

$$= \int_0^x 3(x-t)\sin 5t\,dt$$

Integrating by parts etc.

$$= \frac{3}{5}t - \frac{3}{25}\sin 5t.$$

__Example:__ Find the inverse laplace transform of $\frac{s}{(s+1)(s^2+4)}$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+1)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s+1}}_{F(s)}\cdot\underbrace{\frac{s}{s^2+4}}_{G(s)}\right\}$$

...nes we need to find

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-x} = f(x)$$

$$\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2x = g(x)$$

Hence $\mathcal{L}^{-1}\{F(s)\,G(s)\} = \int_0^x f(x-t)\,g(t)\,d$

$$= \int_0^x e^{-(x-t)} \cos 2t \, dt.$$

If we go back our DE we wanted to find

$$\mathcal{L}^{-1}\{\mathcal{L}\{y(x)\}\} = \mathcal{L}^{-1}\left\{\frac{e^{-s\pi}}{s^2+2^2} - \frac{e^{-2\pi s}}{s^2+2^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\underbrace{\frac{e^{-s\pi}}{2}}_{F_1(s)} \underbrace{\frac{2}{s^2+2^2}}_{G_1(s)}\right\} - \mathcal{L}^{-1}\left\{\underbrace{\frac{e^{-2\pi s}}{2}}_{F_2(s)} \cdot \underbrace{\frac{2}{s^2+2^2}}_{G_2(s)}\right\}$$

Hence we have

$$y(x) = \frac{1}{2}\,\delta_{\pi}(x)\sin 2x - \frac{\delta_{2\pi}(x)\sin 2x}{2}$$ is the desired soln.

Exercise: solve the following DE with Laplace transform.

1) $y'' + y = \delta_{2\pi}(x)\cos x$, $y(0) = 0$, $y'(0) = 1$

2) $y'' - y = 2\sin t$ $\quad y(0) = 2$, $y'(0) = 1$.

3)