Fall 2016 - Math 3410
Final Exam - December 14
Time Limit: 120 Minutes

Name (Print): ____________________

This exam contains 10 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	12	
2	36	
3	36	
4	24	
5	12	
6	0	
Total:	120	

Do not write in the table to the right.

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[1]Exam template credit: http://www-math.mit.edu/~psh

1. Consider the following Laplace's equation in rectangle

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0, \quad 0 < x < 3, \quad 0 < y < 3, \\ u_x(0,y) = -3 \quad \text{and} \quad u_x(3,y) = 0, \\ u_y(x,0) = 3 \quad \text{and} \quad u_y(x,3) = 0 \end{cases}$$

and consider the following function

$$u(x,y) = \frac{1}{2}(x^2 - y^2) - 3x + 3y + 3410.$$

(a) (6 points) Verify that $u(x,y)$ satisfies the Laplace's equation $u_{xx} + u_{yy} = 0$.

(b) (6 points) Verify also that $u(x,y)$ satisfies the given boundary conditions as well (check each of them separately) and conclude that $u(x,y)$ solves the above Laplace's equation with the given boundary conditions.

2. Consider the following Heat conduction problem

$$\begin{cases} 4u_{xx} = u_t, \quad 0 < x < \pi, \quad t > 0, \\ u_x(0,t) = 0 \quad \text{and} \quad u_x(\pi,t) = 0, \quad t > 0, \\ u(x,0) = \cos(3x) + \cos(4x) + \cos(10x), \quad 0 < x < \pi. \end{cases}$$

(a) (6 points) By considering separation of variables $u(x,t) = X(x)T(t)$, rewrite the partial differential equation in terms of two ordinary differential equations in X and T (take arbitrary constant as $-\lambda$).

(b) (4 points) Rewrite the boundary values in terms of X and T. **Be careful on the boundary conditions for u_x** and choose the boundary values which will not give a non-trivial solution and write the ordinary differential equation corresponding to X.

(c) (8 points) Solve the two-point boundary value problem corresponding to X. Find all eigenvalues λ_n and eigenfunctions X_n.

(d) (8 points) For each eigenvalue λ_n you found in (d), rewrite and solve the ordinary differential equation corresponding to T_n.

(e) (5 points) Now write general solution for each n, $u_n(x,t) = X_n(x)T_n(t)$ and find the general solution $u(x,t) = \sum u_n(x,t)$.

(f) (5 points) Using the given initial value and the general solution you found in (f), find the particular solution.

3. Consider the following wave equation which describes the displacement $u(x,t)$ of a piece of flexible string with the initial boundary value problem

$$\begin{cases} 9u_{xx} = u_{tt}, \quad 0 < x < 3, \quad t > 0, \\ u(0,t) = 0 \quad \text{and} \quad u(3,t) = 0, \quad t > 0, \\ u(x,0) = 0 \quad \text{and} \quad u_t(x,0) = 3\pi\sin(\pi x), \quad 0 < x < 3. \end{cases}$$

(a) (6 points) By considering separation of variables $u(x,t) = X(x)T(t)$, rewrite the partial differential equation in terms of two ordinary differential equations in X and T (take arbitrary constant as $-\lambda$).

(b) (4 points) Rewrite the boundary values in terms of X and T and choose the boundary values which will not give a non-trivial solution and then rewrite the ordinary differential equation corresponding to X.

(c) (8 points) Solve the two-point boundary value problem corresponding to X you found in (b). Find all eigenvalues λ_n and eigenfunctions X_n.

(d) (8 points) For each eigenvalue λ_n you found in (d), solve the initial value problem corresponding to T_n.

(e) (5 points) Now write general solution for each n, $u_n(x,t) = X_n(x)T_n(t)$ and find the general solution $u(x,t) = \sum u_n(x,t)$.

(f) (5 points) Using the given initial values, find the particular solution.

4. For the following differential equation

$$xy'' + y' - y = 0$$

(a) (4 points) Find and classify all points as ordinary, regular singular, or irregular singular points.

(b) (5 points) For each of the regular point(s), find the corresponding indicial equation and find the double root r_1 of the indicial equation (Yes there is one double root).

(c) (5 points) Find the corresponding recurrence relation for the root r_1.

(d) (5 points) Find the corresponding power series solution y_1.

(e) (5 points) Use the method of Frobenious and write down the general form of the second solution y_2.

5. Let $f(x)$ be given as

$$f(x) = x, \quad 0 < x < 1.$$

(a) (6 points) Extend $f(x)$ into an even periodic function with period of 2.

(b) (6 points) Find Fourier series $F(x)$.

6. (10 points (bonus)) Solve the first order equation

$$4u_x + u_y = 0$$

with the auxiliary condition

$$u(0, y) = \frac{1}{1+y^2}.$$