Fall 2017 - Math 3410
Practice Exam 1 - Quiz 1 - September 22
Time Limit: 50 Minutes

Name (Print): Solution KEY

This exam contains 8 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	20	
Total:	80	

1. For each part in (a) and (b), write down the letter corresponding to the equation on the list with the specified properties. Also answer parts (c), (d), and (e).

A. $y^{(3410)} + 5y + y' = \cos(y)$.
B. $y' = \frac{6y}{x}$.
C. $y' = 2y^3 - 16$.
D. $(y')^{2016} + 2y^{3410} = 0$.

(a) (4 points) First order linear differential equation which is separable equation.

Solution: B.
We can rewrite the DE in B. as $\frac{dy}{y} = \frac{6}{x}dx$. This is a **first order** and **linear** differential equation which is separable.

(b) (4 points) First order autonomous differential equation.

Solution: C and D. As in C.; $y' = f(y) = 2y^3 - 16$, and f is a function of y only. In D.; $y' = f(x, y) = (-2y^{3410})^{1/2016}$.

(c) (4 points) What is a suitable integrating factor that could be used to solve the linear differential equation you found in part (a)?

Solution: As we can rewrite the DE in B. as $y' - 6/xy = 0$ and from the second homework, we know that if we can write $y' + p(x)y = q(x)$ then

$$\mu(x) = e^{P(x)} = e^{\int p(x)dx} = e^{\int \frac{-6}{x}dx} = e^{-6\log|x|} = x^{-6}$$

for $x > 0$ is an integrating factor for the differential equation in B.

(d) (4 points) Write down the order of the differential equations in (A) to (D).

A. has order 3410 B. has order 1 C. has order 1 D. has order 1

(e) (4 points) Test the DE from (A) to (D) if they are linear or nonlinear.

A. is nonlinear B. is linear C. is nonlinear D. is nonlinear

2. Consider the following differential equation

$$xy^2 + x^2 + (x^2y + y)y' = 0.$$

(a) (3 points) Is the differential equation exact?

Solution: Yes. As $M(x,y) = x^2y + y$ and $N(x,y) = xy^2 + x^2$ and they are both differentable functions everywhere we have

$$\frac{\partial M}{\partial y} = 2xy \qquad \text{and} \qquad \frac{\partial N}{\partial y} = 2xy.$$

Since $M_y = N_x$ then above DE is exact.

(b) (4 points) Find the 1-parameter family of solution of the differential equation (leave the solution as an implicit function).

Solution: We can rewrite the DE as

$$(xy^2 + x^2)dx + (x^2y + y)dy = 0.$$

From this, we want to find a function $f(x,y) = 0$ for which $f_y = x^2y + y$ and $f_x = xy^2 + x^2$. Hence

$$f(x,y) = \int f_y dy = \int (x^2y + y)dy = \frac{1}{2}x^2y^2 + \frac{1}{2}y^2 + g(x).$$

To find $g(x)$ we use the second information $f_x = xy^2 + x^2$;

$$f_x = xy^2 + x^2 = 2xy^2 + g'(x).$$

Hence $g'(x) = x^2$ and $g(x) = x^3/3 + c_1$. Substitute this in f to get

$$f(x,y) = \frac{1}{2}x^2y^2 + \frac{1}{2}y^2 + \frac{x^3}{3} + c_1 = 0$$

and this is an 1-parameter family of solution of the differential equation.

(c) (3 points) Find the particular solution to the initial value problem $y(0) = 2$.

Solution: As the initial value is $y(0) = 2$ means when $x = 0$ then $y = 2$. Therefore,

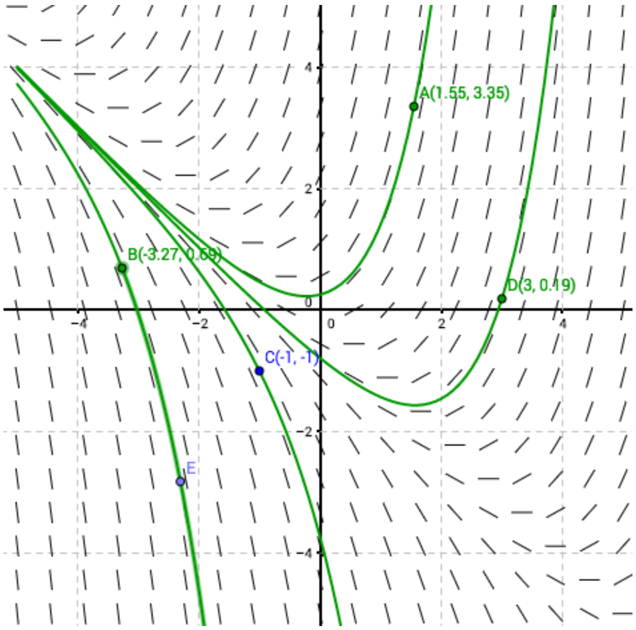
$$f(x,y) = f(0,2) = 0 + 2 + 0 + c_1 = 0$$

which gives us $c_1 = -2$, and

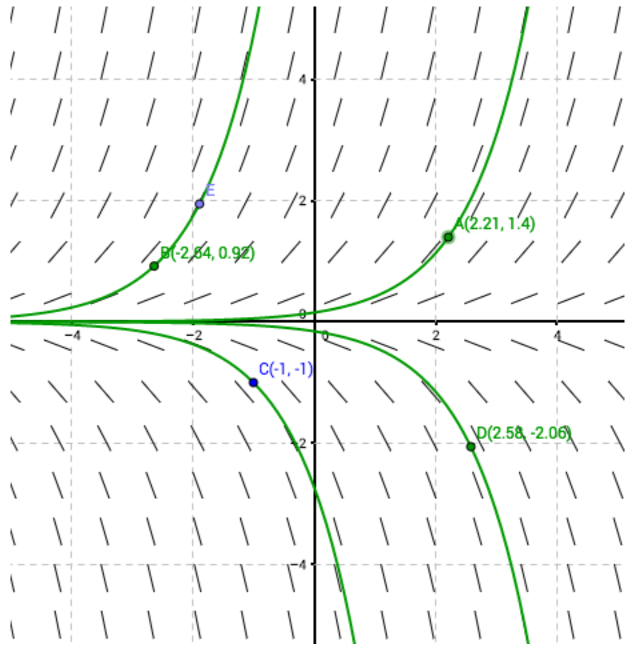
$$f(x,y) = \frac{1}{2}x^2y^2 + \frac{1}{2}y^2 + \frac{x^3}{3} - 2 = 0$$

is the solution to the initial value problem.

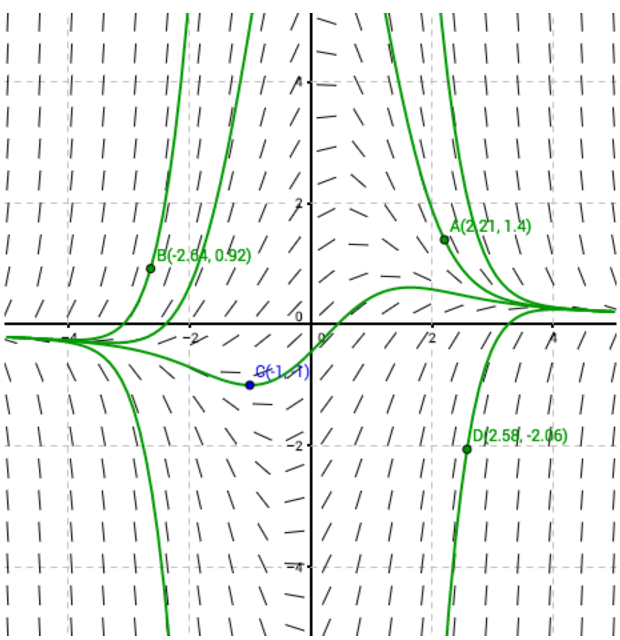
3. Consider the following direction fields;



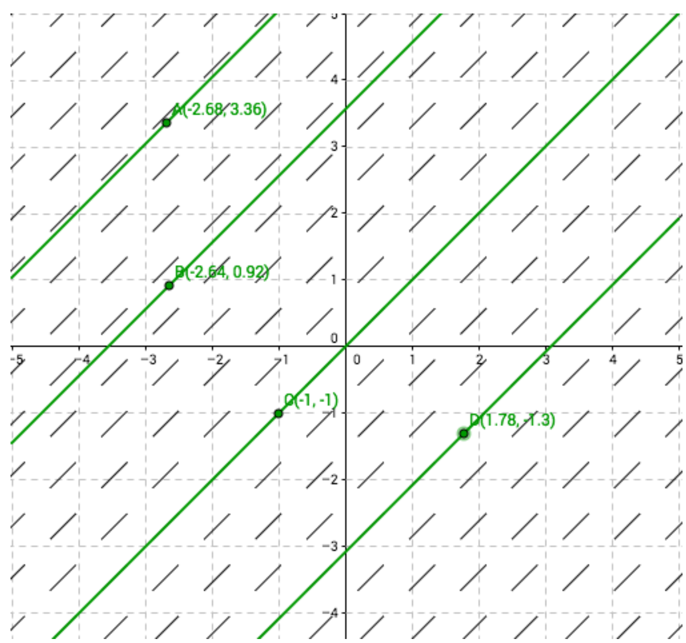
(a) $y' = x + y$



(b) $y' = y$



(c) $y' = 1 - xy$



(d) $y' = 1$

and the differential equations;

1. $y' = y$. 2. $y' = 1 - xy$. 3. $y' = 1$. 4. $y' = x + y$.

(a) (4 points) Match the given direction fields (a) to (d) and differential equations (1) to (4).

(b) (3 points) For each of the direction field (a) to (d), draw the integral curves on the given graph.

(c) (3 points) For each of the direction field (a) to (d), draw the solution with $y(-1) = -1$.

4. (10 points) Find the orthogonal trajectories of the following family of curves.

$$x^2 - y^2 = c^2.$$

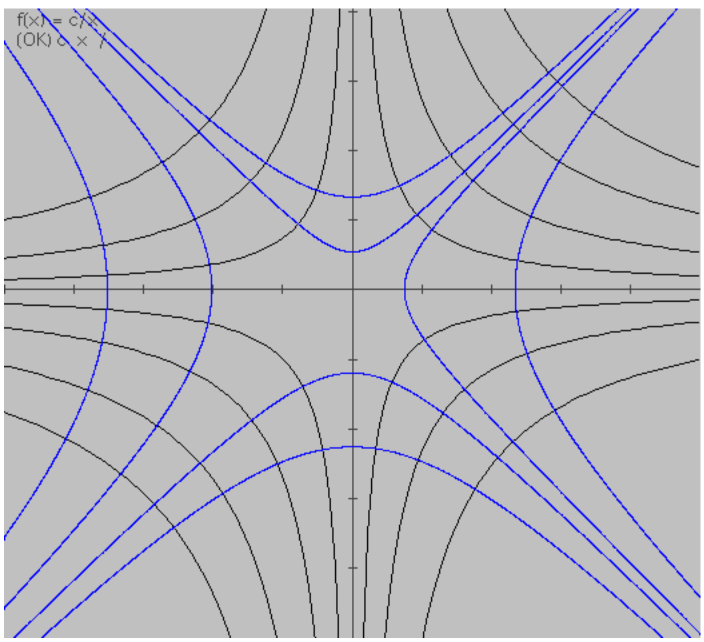
Solution: We are looking for a family of curves which are orthogonal to $x^2 - y^2 = c$. Therefore, we need to find the slope of these curves at every points (x, y). Differentiating gives

$$2x - 2yy' = 0 \quad \text{or} \quad y' = \frac{x}{y} \quad y \neq 0.$$

So above curve has slope x/y at every point (x, y). We are looking for family of curves which is orthogonal. Therefore slope of this family should be $-y/x$ for $x \neq 0$. As x/y times $-y/x$ is -1. So, the new family of curves has slope $y' = -y/x$. Therefore, we need to solve the DE $y' = -y/x$. This is a separable DE and 1-parameter family of solution is

$$\log y = -\log x + \log c \quad \text{equivalently} \quad y = \frac{c_1}{x} \; x \neq 0, \, y \neq 0$$

is the family of curves we are looking for.



In the graph, the blue curves are $x^2 - y^2 = c$ and black ones are $xy = c$ (I do not expect you to draw this).

5. (10 points) Consider the initial value problem

$$(t^2-4)y' + \frac{t+2}{t}y = \frac{t^3}{t-5}, \quad y(4) = \frac{1}{2}.$$

Without solving the equation, what is the largest interval for t in which a unique solution is guaranteed to exist?

Solution: Here we are looking for an interval which contains 4. First, as we have t and $t-5$ in the denominator, $t \neq 5, 0$. Moreover, coefficient of y' should not be zero, from which we get $t \neq +2, -2$. Now, I can rewrite the DE (by dividing t^2-4)

$$y' + \frac{t+2}{(t^2-4)t}y = \frac{t^3}{(t-5)(t^2-4)}$$

Hence this is a linear first order DE with

$$p(t) = \frac{t+2}{(t^2-4)t} \qquad \text{and} \qquad q(t) = \frac{t^3}{(t-5)(t^2-4)}.$$

Now $p(t)$ is continuous except at $t = 0, 2$ and $q(t)$ is continuous except at $5, 2, -2$.

Therefore, the largest interval containing 4 and does not contain any discontinuity point is $(2, 5)$. By the existence uniqueness theorem for linear first order DE, we know that there exists a unique solution to above DE with any initial value for $t \in (2, 5)$.

6. Consider the autonomous equation

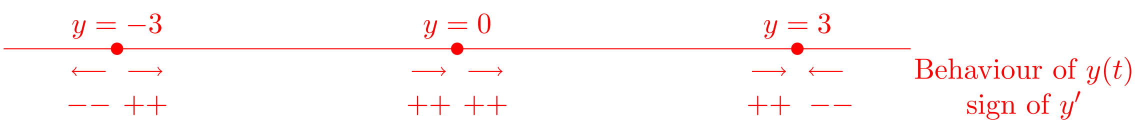
$$y' = 9y^2 - y^4 = y^2(3-y)(3+y).$$

(a) (4 points) Find all equilibrium solutions.

Solution: Here $f(y) = y^2(3-y)(3+y) = 0$ has solutions $y = 0$, $y = 3$, $y = -3$ are the all equilibrium solutions.

(b) (6 points) Classify the stability of each equilibrium solution. Justify your answer.

Solution:



Therefore, (close) solutions above $y = 3$ equilibrium solution approaches to $y = 3$, similarly solutions below $y = 3$ equilibrium solution approach to $y = 3$. Therefore, $y = 3$ is stable (or asymptotically stable) equilibrium solution. Solutions above $y = 0$ equilibrium solution run away from $y = 0$ and solutions below $y = 0$ equilibrium solution approach to $y = 0$ equilibrium solution. Therefore, $y = 0$ semi-stable equilibrium solution. Solutions above $y = -3$ equilibrium solution runaway from $y = -3$ and similarly solutions below $y = -3$ equilibrium solution also runaway from $y = -3$. Hence $y = -3$ is an unstable equilibrium solution.

(c) (3 points) If $y(22/7) = \pi$, what is $\lim_{t\to\infty} y(t)$? (Hint: $\pi = 3.14159265...$)

Since $y(22/7) = \pi$ and $y_0 = \pi > 3$ and $y = 3$ is a stable solution

$$\lim_{t\to\infty} y(t) = 3.$$

(d) (3 points) If $y(2\pi) = -3$, what is $y(t)$?

Since $y(t) = -3$ is an equilibrium solution then for every t, $y(t) = -3$.

(e) (4 points) If $y(4) = \lambda$. Find intervals for λ for which $\lim_{t\to\infty} y(t) = 0$.

Note that $y = 0$ is a semi-stable equilibrium solution therefore, for positive values of y_0 but close to zero, the solutions will approach to $y = 3$ stable equilibrium solutions. But for $-3 < y_0 < 0$, then all solutions will approach to $y = 0$ is an equilibrium solution. Therefore, $\lambda \in (-3, 0]$. (Notice that $y = 0$ is solution and $y = 0$ has limit 0 as $t \to \infty$.)

Indeed, one can verify above results by checking the gradient vector field for above autonomous differential equation;