UCONN - Math 3410 - Fall 2017 - Quiz 3

Name: Solution KEY

Question: Consider the differential equation (k is some constant)

$$y'' + k^2 y = 0, \quad y(0) = 1,\ y'(0) = 2.$$

(a) You are going to find a power series solution to above differential equation around $x_0 = 0$. As a first step, using power series method, find the recurrence relation. By checking some of the terms, try to find a pattern.

Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Plug into the differential equation, one gets

$$y'' + k^2 y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} k^2 a_n x^n = 0.$$

If we change index n in the second summation we get

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=2}^{\infty} k^2 a_{n-2} x^{n-2} \\ &= \sum_{n=2}^{\infty} [n(n-1) a_n + k^2 a_{n-2}] x^{n-2}. \end{aligned}$$

Since this is true for every x, coefficients of x^{n-2} should be zero for every $n \geq 2$. Therefore,

$$n(n-1) a_n + k^2 a_{n-2} = 0 \quad \text{equivalently} \quad a_n = \frac{-k^2 a_{n-2}}{n(n-1)} \quad \text{for} \quad n \geq 2.$$

Now at this point, check the even terms first (using $y(0) = a_0 = 1$)

$$\begin{aligned} a_2 &= \frac{-k^2 a_0}{2} = \frac{-k^2}{2} \\ a_4 &= \frac{-k^2 a_2}{4\,3} = \frac{k^4}{4\,3\,2} = \frac{k^4}{4!} \\ a_6 &= \frac{-k^2 a_4}{6\,5} = \frac{-k^6 a_0}{6\,5\,4!} = \frac{-k^6}{6!} \\ &\vdots \\ a_{2n} &= \frac{(-1)^n k^{2n}}{2n!} \quad \text{for} \quad n \geq 1. \end{aligned}$$

Similarly, odd terms are (using $y'(0) = a_1 = 2$)

$$\begin{aligned}
a_3 &= \frac{-k^2 a_1}{3\,2} = \frac{-2k^2}{3\,2} \\
a_5 &= \frac{-k^2 a_3}{5\,4} = \frac{k^4 a_1}{5\,4\,3!} = \frac{2k^4}{5!} \\
a_7 &= \frac{-k^2 a_5}{7\,6} = \frac{-k^6 a_1}{7\,6\,5!} = \frac{-2k^6}{7!} \\
&\vdots \\
a_{2n+1} &= \frac{(-1)^n k^{2n} a_1}{(2n+1)!} = \frac{(-1)^n 2k^{2n+1}}{(2n+1)!\,k} \quad \text{for} \quad n \geq 1.
\end{aligned}$$

(b) Using part (a) find the power series solution to the above differential equation. (Hint: combine even and odd terms).

Since

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n = \text{even terms} + \text{odd terms} \\
&= \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}.
\end{aligned}$$

Now using part (b) we can write

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n k^{2n}}{2n!} x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n 2k^{2n+1}}{(2n+1)!\,k} x^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (kx)^{2n}}{2n!} + \frac{2}{k} \sum_{n=0}^{\infty} \frac{(-1)^n (kx)^{2n+1}}{(2n+1)!}
\end{aligned}$$

which is the power series solution we are looking for.

(c) Find the function representations of the power series you found in (b).

From part (c) we have

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n (kx)^{2n}}{2n!} + \frac{2}{k} \sum_{n=0}^{\infty} \frac{(-1)^n (kx)^{2n+1}}{(2n+1)!} \\
&= \cos(kx) + \frac{2}{k} \sin(kx).
\end{aligned}$$