Fall 2017 - Math 3410
Exam 1 - October 4
Time Limit: 50 Minutes

Name (Print): ____________________

This exam contains 6 pages (including this cover page) an empty scratch paper and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	10	
3	10	
4	10	
5	20	
6	10	
Total:	80	

Do not write in the table to the right.

[1]

[1] Exam template credit: `http://www-math.mit.edu/~psh`

1. For each part in (a) and (b), write down the letter corresponding to the equation on the list with the specified properties. Also answer parts (c), (d), and (e).

   A. $y' + \sin(y) = \cos(y)$.

   B. $y'' + \sin(x)\, y' + \cos(x)\, y = \tan(x)$.

   C. $y' + \frac{1}{x} y = e^{2x}$.

   D. $y^{(7)} + 25(y')^{16} + y'' + y + x = 0$.

   (a) (4 points) First order linear differential equation (has only one answer).
   Solution: C.

   (b) (4 points) First order autonomous differential equation (has only one answer).
   Solution A.

   (c) (4 points) What is a suitable integrating factor that could be used to solve the linear differential equation you found in part (a)?
   Solution: An integrating factor $\mu(x)$ is

$$\mu(x) = e^{\int \frac{1}{x} dx} = x.$$

   (d) (4 points) Write down the order of the differential equations in (A) to (D).

   A. has order 1 B. has order 2 C. has order 1 D. has order 7.

   (e) (4 points) Test the differential equations from (A) to (D) if they are linear or nonlinear.

   A. is Nonlinear B. is linear C. is linear D. is Nonlinear.

2. Consider the following differential equation

$$2y' + y = e^x.$$

(a) (5 points) Find the 1-parameter family of solution of the differential equation. Write your solution in *explicit* form. (i.e., solve for y).

Solution: There are many ways you can solve this. Once you find that $\mu(x) = e^{x/2}$ is an integrating factor then by multiplying the DE with μ and dividing by 2 you can rewrite the DE as

$$e^{x/2}y' + \frac{1}{2}e^{x/2}y = \frac{1}{2}e^{3x/2}.$$

This can be written as

$$(e^{x/2}y)' = \frac{1}{2}e^{3x/2} \quad \text{hence} \quad e^{x/2}y = \int \frac{1}{2}e^{3x/2}dx.$$

From this you should get

$$e^{x/2}y(x) = \frac{1}{2}\frac{2}{3}e^{3x/2} + c \quad \text{or} \quad y(x) = \frac{1}{3}e^x + ce^{-x/2}.$$

(b) (2 points) Find the solution of the differential equation with the given initial value $y(0) = \alpha$.

Solution: Since $y(0) = \alpha$, we should have

$$y(0) = \alpha = \frac{1}{3} + c \quad \text{or} \quad c = \alpha - \frac{1}{3}.$$

Therefore, the particular solution is

$$y(x) = \frac{1}{3}e^x + (\alpha - \frac{1}{3})e^{-x/2}.$$

(c) (3 points) For what value(s) of α, the solution you found in (b) remains finite as $x \to -\infty$?

Since $e^x \to 0$ as $x \to -\infty$, and $e^{-x/2} \to \infty$ as $x \to -\infty$ we should choose $\alpha = 1/3$ so that the solution remains bounded. For any other values of α , the solution goes to infinity.

3. (10 points) Find the orthogonal trajectories of the following family of lines passing through the point $(1,1)$.

   Note that the family of curves passing through $(1,1)$ has equation

   $$(y-1) = c(x-1) \quad \text{wherer } c \text{ is the parameter.}$$

   Hence

   $$y' = c = \frac{y-1}{x-1}.$$

   This is the slope of the above family. The family of curves will be orthogonal to this hence the slope should -1/slope of above family

   $$y' = -\frac{x-1}{y-1} \quad \text{or} \quad (y-1)y' = -(x-1).$$

   If you solve the DE you get

   $$\frac{y^2}{2} - y = -(\frac{x^2}{2} - x) + r^2$$

   Equivalently,

   $$(y-1)^2 + (x-1)^2 = r^2.$$

4. (10 points) Find the most general function $N(x,y)$ so that the equation

   $$(\cos(x)\sin(y) - y^2)dx + N(x,y)dy = 0$$

   is exact.

   Now here the first step should be trying to $f(x,y) = 0$ such that $f_x = (\cos(x)\sin(y) - y^2)$ and $f_y = N$. Integrating f_x with respect to x, one gets

   $$f(x,y) = \sin(x)\sin(y) - xy^2 + h(y) = 0 \quad \text{for any differentiable function } h.$$

   Then since $f_y = N$ we get

   $$N = \sin(x)\cos(y) - 2xy + h'(y) \quad \text{for any differentiable function } h.$$

   This is the most general function N.

5. Consider the autonomous equation

$$y' = (y+2)^2(1-y)^2.$$

(a) (4 points) Find all equilibrium solutions.

$y' = 0$ when $y = 1$ and $y = -2$. Hence $y = 1, -2$ are the equilibrium solutions.

(b) (6 points) Classify the stability of each equilibrium solution. Justify your answer.

Since $y' > 0$ as the right hand-side above is always positive we have all the slopes pointing up. Hence any solution will go up until it touches to the equilibrium solution. From this one gets both equilibrium solutions are semi-stable.

(c) (3 points) If $y(-2) = \mathbf{e}$, what is $\lim_{t\to\infty} y(t)$? Briefly explain your answer.
(Hint: $\mathbf{e} = 2.71828...$)

Since e > 1 then $y(t) \to \infty$ as $t \to \infty$.

(d) (3 points) If $y(1) = 0$, what is $\lim_{t\to\infty} y(t)$?

Since $y = 1$ is a semi-stable equilibrium solution $\lim_{t\to\infty} y(t) = 1$.

(e) (4 points) If $y(0) = \lambda$. Find intervals for λ for which $\lim_{t\to\infty} y(t) = 1$.

Since $y(t) = 1$ is a semi-stable solution then for any values of λ in $(-2, 1]$ the solutions will approach to $y = 1$.

6. (10 points) Using Laplace transform method solve the following DE

$$y'' - 2y' + y = e^{-x} \quad \text{with } y(0) = 0 \; y'(0) = 0.$$

Solution: Since Now if we take the Laplace transform of both sides we get

$$\mathcal{L}\{y'' - 2y' + y\} = \mathcal{L}\{e^{-x}\} \quad \text{or} \quad \mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{e^{-x}\}$$

where we have used the property (1) from the notes, namely, the linearity. Now the using properties (2) and (3) from the lecture notes we get

$$s^2\mathcal{L}\{y(x)\} - sy(0) - y'(0) - 2s\mathcal{L}\{y(x)\} + 2y(0) + \mathcal{L}\{y(x)\} = \frac{1}{s+1}.$$

Solving for $\mathcal{L}\{y(x)\}$ we get

$$(s^2 - 2s + 1)\mathcal{L}\{y(x)\} = \frac{1}{s+1} \quad \text{or} \quad \mathcal{L}\{y(x)\} = \frac{1}{s+1}\frac{1}{(s-1)^2}.$$

Since

$$\frac{1}{s+1}\frac{1}{(s-1)^2} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

we need to find A, B, C. After some algebra, one can find $= 1/4, B = -1/4, C = 1/2$ and hence we have

$$\mathcal{L}\{y(x)\} = \frac{1}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2}.$$

Using the inverse Laplace transform on both sides we get

$$\mathcal{L}^{-1}\mathcal{L}\{y(x))\} = \mathcal{L}^{-1}\{\frac{1}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2}\}.$$

Hence using linearity we have

$$y(x) = \frac{1}{4}\mathcal{L}^{-1}\{\frac{1}{s+1}\} - \frac{1}{4}\mathcal{L}^{-1}\{\frac{1}{s-1}\} + \frac{1}{2}\mathcal{L}^{-1}\{\frac{1}{(s-1)^2}\}\}.$$

This gives us

$$y(x) = \frac{1}{4}e^{-x} - \frac{1}{4}e^{x} + x\frac{1}{2}e^{x}.$$

SCRATCH PAPER