Fall 2017 - Math 3410
Exam 2 - November 3
Time Limit: 50 Minutes

Name (Print): Solution KEY

This exam contains 8 pages (including this cover page) an empty scratch paper and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books or notes on this exam.

You are required to show your work on each problem on this exam.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	12	
3	12	
4	12	
5	24	
6	0	
Total:	80	

[1] Exam template credit: `http://www-math.mit.edu/~psh`

1. Consider the following differential equation

$$y'' - y = 0 \quad \text{with} \quad y(0) = 2 \text{ and } y'(0) = 0.$$

(a) (3 points) Classify all points as ordinary, regular singular, or irregular singular points. : Since all coefficients are 1, they are all smooth. Therefore, all points are ordinary points.

(b) (7 points) You are going to find a power series solution around $x_0 = 0$. As a first step, using power series method, find the recurrence relation. Show your work!

Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Using the DE and initial values we will find a_n. Since $y(0) = 2$ then we have $a_0 = 2$ and since $y'(0) = 0$ we also have $a_1 = 0$. We need to find all remaining a_n's. To this end, we need to find $y''(x)$;

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Using this in the DE we have

$$\begin{aligned} 0 = y'' - y &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=2}^{\infty} a_{k-2} x^{k-2} \\ &= \sum_{k=2}^{\infty} [k(k-1)a_k - a_{k-2}] x^{k-2}. \end{aligned}$$

Note that $y(0) = 2$ hence $y(0) = a_0 = 2$. Also, $y'(0) = 0 = a_1$. We have the recurrence relation

$$k(k-1)a_k - a_{k-2} \quad k = 2, 3, \ldots.$$

Or, equivalently, the recurrence relation we are looking for is

$$a_k = \frac{a_{k-2}}{k(k-1)} \quad k = 2, 3, 4, \ldots.$$

$$\begin{aligned} \text{for } k = 2, \quad a_2 &= \frac{a_0}{2 \times 1} = \frac{a_0}{2!} = \frac{2}{2!} \\ \text{for } k = 3, \quad a_3 &= \frac{a_1}{3 \times 2} = \frac{a_1}{3!} = 0 \\ \text{for } k = 4, \quad a_4 &= \frac{a_2}{4 \times 3} = \frac{a_0}{4!} = \frac{2}{4!} \\ \text{for } k = 5, \quad a_5 &= \frac{a_3}{5 \times 4} = \frac{a_1}{5!} = 0 \\ &\vdots \end{aligned}$$

(c) (6 points) Using part (b) find the power series solution to the above differential equation. (Hint: combine even and odd terms).

we have realized from part (b) that

$$a_{2n} = \frac{2}{(2n)!} \quad \text{and} \quad a_{2n-1} = \frac{a_1}{(2n-1)!} = 0.$$

Now

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \text{even terms} + \text{odd terms} = \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=1}^{\infty} a_{2n-1} x^{2n-1}$$
$$= \sum_{n=0}^{\infty} \frac{2}{(2n)!} x^{2n}.$$

(d) (4 points) Using part (c) and directly solving the DE by finding the characteristic equation, show that

$$\frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

Since the DE equation has characteristic $r^2 - 1 = 0$ which has roots $r = \pm 1$. Therefore the general solution is

$$y(x) = c_0 e^x + c_1 e^{-x}.$$

Using the initial conditions one can observe that $c_0 = c_1 = 1$. Hence $y(x) = e^x + e^{-x}$. Using part (c) we have

$$e^x + e^{-x} = 2 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad \text{or} \quad \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

2. Consider the following differential equation

$$y'' + 4(y^2 + 1)y' + xy = 0.$$

(a) (9 points) Use the **second method** to find first four terms of the power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ around $x_0 = 0$ with the initial conditions $y(0) = 0$ and $y'(0) = 1$.

Note that $y(x)$ has Taylor series expansion around $x_0 = 0$. Hence,

$$y(x) = \sum_{n=0}^{\infty} \frac{y^n(0)}{n!} x^n = \sum_{n=0}^{\infty} a_n x^n$$

To find a_0, a_1, a_2, a_3 we need to find $y(0) = a_0, y'(0) = a_1, y''(0) = 2a_2, y'''(0) = 6a_3$. As $y(0) = 0 = a_0$ and $y'(0) = 1 = a_1$. We next find $y''(0)$ by using the DE.

$$y''(0) + 4(y^2(0) + 1)y'(0) + 0y(0) = 0.$$

We get $y''(0) = -4$. From this we get $a_2 = -2$. We next find $y'''(0)$ and to do this we first need to do implicit differentiation in the DE;

$$y''' + 4(y^2 + 1)y'' + 8yy'y' + y + xy' = 0.$$

We then evaluate this at $x = 0$;

$$y'''(0) + 4(y^2(0) + 1)y''(0) + 8y(0)y'(0)y'(0) + y(0) + 0y'(0)$$

and get $y'''(0) = 16$. Hence $a_3 = 16/6$

$a_0 = 0$ $\quad a_1 = 1$ $\quad a_2 = -2$ $\quad a_3 = 16/6.$

(b) (3 points) Using your above work, write the first four terms of the solution $y(x)$

$$y(x) = 0 + x - 2x^2 + \frac{16}{6}x^3 + \dots$$

3. (12 points) Classify all the points as ordinary point, regular singular point, or irregular singular point for the following differential equation

$$(x+1)^3y'' + (x+1)y' + 4(x+1)y = 0.$$

Solution: If we rewrite the DE we get

$$y'' + \frac{(x+1)}{(x+1)^3}y' + \frac{4(x+1)}{(x+1)^3}y = 0.$$

Hence $P(x) = \frac{(x+1)}{(x+1)^3}$ and $Q(x) = \frac{4(x+1)}{(x+1)^3}$. From this we observe that $P(x)$ and $Q(x)$ are analytic at every point except at $x = -1$. Hence all points are ordinary points except $x = -1$ and $x = -1$ is a singular point.

To classify $x = -1$ we need to check if it is regular singular or irregular singular point. To this end, we need to check

$$\lim_{x\to -1}(x+1)P(x) = \lim_{x\to -1}(x+1)\frac{(x+1)}{(x+1)^3} = \text{does not exist.}$$

Hence $x = -1$ is a irregular singular point.

4. Consider the following differential equation

$$x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0.$$

(a) (6 points) Find the indicial equation corresponding to the regular singular point $x_0 = 0$. (Do not try to solve the differential equation). Solution: Since we just want to find indicial equation we can test it with $\phi(x) = x^r$ and then we collect the terms with lowest power of x.

$$x^2r(r-1)x^{r-2} + xrx^{r-1} + x^2x^r - \frac{1}{4}x^r = r(r-1)x^r + rx^r + x^{r+2} - \frac{1}{4}x^r.$$

Hence the lowest power of x is r and the coefficient is $r(r-1) + r - \frac{1}{4} =$ is the indicial equation;

$$r^2 - \frac{1}{4} = 0.$$

(b) (6 points) Write the general form of the solution(s).
Solution: Since the indicial equation has roots $r = \pm 1/2$ and $r_1 = 1/2$, $r_2 = -1/2$ we have $r_1 - r_2 = 1$ which is integer. By the Method of Frobenious we have

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$

and

$$y_2(x) = x^{r_2}\sum_{n=0}^{\infty} b_n x^n + C\ln|x|y_1(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}} + C\ln|x|y_1(x).$$

5. Consider the function $f(x) = x^2$ on $(-\pi, \pi)$ and $f(x + 2\pi) = f(x)$.

   (a) (3 points) What is the period $2L$ of $f(x)$?
   Solution: $2L = 2\pi$ is the period of f. For later use $L = \pi$.

   (b) (3 points) Is $f(x)$ an odd or even function? Show your work.
   Solution: Since $f(x) = x^2 = f(-x)$, f is an even function.

   (c) (5 points) Find the sine terms of the Fourier series of $f(x)$. Solution: Since f is even function all sine terms are zero. Hence

$$b_n = 0 \text{ for all } n = 1, 2, \ldots.$$

   (d) (5 points) Find the cosine terms (including a_0) of the Fourier series of $f(x)$.
   Solution: We first find a_0;

$$a_0 = \frac{1}{\pi}\int_{\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{\pi}^{\pi} x^2 dx = \frac{1}{\pi}x^3/3|_{x=-\pi}^{x=\pi} = \frac{2}{3}\pi^2.$$

   Next

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(\frac{n\pi x}{\pi})dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)dx = \frac{4(-1)^n}{n^2}.$$

(e) (4 points) Write the Fourier series $F(x)$ of the function $f(x)$.

Solution: Hence we get

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx)$$

(f) (4 points) Using Fourier series convergence theorem, check the points where $F(x)$ and $f(x)$ agree and do not agree.

Solution: Since $f(x)$ is continuous on $(-\pi, \pi)$ we have $F(x) = f(x)$. At $x = \pi$ and $x = -\pi$, f is not defined however, $F(x) = \pi^2$ at those points.

6. (10 points (bonus)) Use part (e) in Question 5 and verify that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Solution: Since $F(x) = f(x)$ at $x = 0$, we have

$$0 = f(0) = F(0) = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(0) = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}.$$

Hence

$$-\frac{1}{3}\pi^2 = \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \quad \text{or} \quad \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

SCRATCH PAPER