UNIVERSITY OF CONNECTICUT

Fall 2017 - Math 3410
Final Exam - December 13
Time Limit: 120 Minutes

Name (Print): ______________________________

This exam contains 8 pages (including this cover page) an empty scratch paper and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books or notes on this exam.

You are required to show your work on each problem on this exam.

Do not write in the table to the right.

Problem	Points	Score
1	12	
2	32	
3	32	
4	12	
5	12	
6	20	
7	0	
Total:	120	

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[1] Exam template credit: `http://www-math.mit.edu/~psh`

1. Consider the following partial differential equation with boundary conditions

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0, \quad 0 < x < 1, \quad 0 < y < \pi, \\ u(0,y) = \sin(y) + 3410 \quad \text{and} \quad u(1,y) = e\sin(y) + 3410, \\ u_y(x,0) = e^x \quad \text{and} \quad u_y(x,\pi) = -e^x \end{cases}$$

and the following function

$$u(x,y) = e^x \sin(y) + 3410. \tag{1}$$

(a) (6 points) Verify that $u(x,y)$ satisfies the Laplace's equation $u_{xx} + u_{yy} = 0$.
Solution: Since

$$u_{xx} = e^x \sin y \quad \text{and} \quad u_{yy} = -e^x \sin y$$

we get $u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$.

(b) (6 points) Verify also that $u(x,y)$ given in (1) satisfies the given boundary conditions as well (check each of them separately) and conclude that $u(x,y)$ in (1) solves the above Laplace's equation with the given boundary conditions.
Solution: Since $u(x,y) = e^x \sin y$ then we have

$$u(0,y) = \sin y + 3410 \quad \text{and} \quad u(1,y) = e \sin y + 3410.$$

On the other hand, as $u_y = e^x \cos(y)$ we then have

$$u_y(x,0) = e^x \quad \text{and} \quad u_y(x,\pi) = -e^x.$$

These show that $u(x,y)$ given in (1) solves above Laplace's equation with boundary conditions.

2. Consider the following Heat conduction problem

$$\begin{cases} u_{xx} = u_t, & 0 < x < 2, \quad t > 0, \\ u(0,t) = 0 \quad \text{and} \quad u(2,t) = 0, & t > 0, \\ u(x,0) = 3\sin(\pi x) - 4\sin(\frac{3\pi x}{2}), & 0 < x < 2. \end{cases}$$

(a) (6 points) By considering separation of variables $u(x,t) = X(x)T(t)$, rewrite the partial differential equation in terms of two ordinary differential equations in X and T (take arbitrary constant as $-\lambda$).
Solution: Rewrite the PDE as $u_{xx} - u_t = 0$. Let $u(x,t) = X(x)T(t)$. Then

$$u_{xx} = X''T \quad \text{and} \quad u_t = XT'.$$

Substitute this in to the differential equation to get

$$u_{xx} - u_t = X''T - XT' = 0 \quad \text{equivalently} \quad \frac{X''}{X} = \frac{T'}{T} = -\lambda.$$

Hence

$$\frac{X''}{X} = -\lambda \quad \to \quad X'' + \lambda X = 0,$$
$$\frac{T'}{T} = -\lambda \quad \to \quad T' + \lambda T = 0.$$

(b) (4 points) Rewrite the boundary values in terms of X and T and choose the boundary values which will give a non-trivial solution and write the ordinary differential equation corresponding to X.
Solution: We have at $x = 0$

$$u(0,t) = X(0)T(t) = 0; \quad \text{one has either} \quad X(0) = 0 \quad \text{or} \quad T(t) = 0.$$

At $x = 2$

$$u(2,t) = X(2)T(t) = 0; \quad \text{one has either} \quad X(2) = 0 \quad \text{or} \quad T(t) = 0.$$

We know that the choice of $T(t) = 0$ gives only the trivial solution as $u(x,t) = X(x)T(t) = 0$.
Therefore, we choose our boundary conditions as $X(0) = 0$ and $X(2) = 0$ in order to obtain the non-trivial solution. Now if we rewrite the ordinary differential equation corresponding to X we get

$$X'' + \lambda X = 0, \quad X(0) = 0 \quad \text{and} \quad X(2) = 0.$$

(c) (6 points) Solve the two-point boundary value problem corresponding to X. Find all eigenvalues λ_n and eigenfunctions X_n.
Solution: For $\lambda = 0$, you can check that $X(x) = ax + b$ is the general solution and using the boundary conditions we get $X(0) = b = 0$ and $X(2) = 2a = 0$. Hence $X = 0$ is the only solution which is trivial solution.

For $\lambda < 0$, in this case $X(x) = C_0 e^{\sqrt{-\lambda}x} + C_1 e^{\sqrt{-\lambda}x}$ is the general solution and using $X(0) = 0$ we have $C_0 = -C_1$ and using $X(2) = 0$ we get $C_0 = C_1 = 0$. Hence we get trivial solution.
For $\lambda > 0$ We know that the non-trivial general solutions is

$$X(x) = A\cos(kx) + B\sin(kx)$$

where we choose $\lambda = k^2$, $k > 0$. Using this general solution and the first boundary condition

$$X(0) = A\cos(0) + B\sin(0) = A = 0 \quad \text{therefore we have} \quad A = 0.$$

Using the second boundary condition, we get (as $A = 0$, $X(x) = B\sin(kx)$)

$$X(2) = B\sin(2k) = 0.$$

This holds when $2k = n\pi$ for $n = 1, 2, \ldots$. Hence we get $k = n\pi/2$, or equivalently, we get the eigenvalues

$$\lambda_n = k^2 = \frac{n^2\pi^2}{2^2} \quad \text{for } n = 1, 2, \ldots.$$

The corresponding eigenfunction (corresponding to λ_n) is

$$X_n(x) = \sin(\frac{n\pi x}{2}) \quad \text{for } n = 1, 2, \ldots.$$

(d) (6 points) For each eigenvalue λ_n you found in (d), rewrite and solve the ordinary differential equation corresponding to T_n.
Solution: Since the ordinary differential equation corresponding to T is

$$T' + \lambda T = 0.$$

Plug in $\lambda_n = k^2 = \frac{n^2\pi^2}{2^2}$ we get (for each n we have a different solution T_n)

$$T_n' + \frac{n^2\pi^2}{2^2} T_n = 0.$$

We know that this is a first order linear ordinary differential equation and its solution is

$$T_n(t) = C_n e^{-\frac{n^2\pi^2}{2^2}t} \quad \text{for } n = 1, 2, \ldots$$

for some C_n.

(e) (2 points) Now write general solution for each n, $u_n(x,t) = X_n(x)T_n(t)$ and find the general solution $u(x,t) = \sum u_n(x,t)$.
Solution: We know that the solution for each n is

$$u_n(x) = X_n(x)T_n(t) = \sin(\frac{n\pi x}{2}) C_n e^{-\frac{n^2\pi^2}{2^2}t} \text{ for } n = 1, 2, \ldots, \quad \text{and.}$$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi x}{2}) e^{-\frac{n^2\pi^2}{2^2}t}.$$

(f) (8 points) Using the given initial value and the general solution you found in (f), find the particular solution.
Solution: Using the given initial condition $u(x,0) = 3\sin(\pi x) - 4\sin(\frac{3\pi x}{2})$ and evaluating the solution at $t = 0$ we found in the previous part we get

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi x}{2}) = 3\sin(\pi x) - 4\sin(\frac{3\pi x}{2}).$$

From this we see that $C_2 = 3$ and $C_3 = -4$ and all other $C_n = 0$. The solution we are looking for is

$$u(x,t) = 3\sin(\pi x) e^{-\frac{2^2\pi^2}{2^2}t} - 4\sin(\frac{3\pi x}{2}) e^{-\frac{3^2\pi^2}{2^2}t}.$$

3. Consider the following wave equation which describes the displacement $u(x,t)$ of a piece of flexible string with the initial boundary value problem

$$\begin{cases} 9u_{xx} = u_{tt}, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0, \quad t > 0, \\ u(x,0) = 3435 sin(\pi x) + 2018\sin(5\pi x) \quad \text{and} \quad u_t(x,0) = 0, \quad 0 < x < 1. \end{cases}$$

(a) (6 points) By considering separation of variables $u(x,t) = X(x)T(t)$, rewrite the partial differential equation in terms of two ordinary differential equations in X and T (take arbitrary constant as $-\lambda$).

Solution: Rewrite the differential equation as $9u_{xx} - u_{tt} = 0$. Let $u(x,t) = X(x)T(t)$. Then we get

$$u_{xx} = X''T \quad \text{and} \quad u_{tt} = XT''.$$

Substitute this into the partial differential equation $9u_{xx} - u_{tt} = 0$ to get

$$9u_{xx} - u_{tt} = 9X''T - XT'' = 0.$$

Dividing by $9XT$ we get

$$\frac{X''}{X} = \frac{T''}{9T}.$$

Notice that the left-hand side is a function of x only and the right-hand side is function of t only and as they are same, this is possible only if they are the same constant;

$$\frac{X''}{X} = \frac{T''}{9T} = -\lambda.$$

From this we get

$$\begin{aligned} \frac{X''}{X} = -\lambda \quad &\rightarrow \quad X'' + \lambda X = 0 \\ \frac{T''}{9T} = -\lambda \quad &\rightarrow \quad T'' + 9\lambda T = 0. \end{aligned}$$

(b) (4 points) Rewrite the boundary values in terms of X and T and choose the boundary values which will give a non-trivial solution and then rewrite the ordinary differential equation corresponding to X.

Solution: Since $u(x,t) = X(x)T(t)$ we have when $x = 0$

$$u(0,t) = X(0)T(t) = 0 \quad \text{we should have either} \quad X(0) = 0 \quad \text{or} \quad T(t) = 0.$$

At $x = 1$, we have

$$u(1,t) = X(1)T(t) = 0 \quad \text{we should have either} \quad X(1) = 0 \quad \text{or} \quad T(t) = 0.$$

We know that $T(t) = 0$ will give only the trivial solution. Therefore, we should choose

$$X(0) = 0 \quad \text{and} \quad X(1) = 0.$$

Then X satisfies the following two-point boundary condition

$$X'' + \lambda X = 0, \quad X(0) = 0 \quad \text{and} \quad X(1) = 0.$$

(c) (6 points) Solve the two-point boundary value problem corresponding to X you found in (b). Find all eigenvalues λ_n and eigenfunctions X_n.

Solution: For $\lambda = 0$, you can check that $X(x) = ax + b$ is the general solution and using the boundary conditions we get $X(0) = b = 0$ and $X(1) = a = 0$. Hence $X = 0$ is the only solution which is trivial solution.

For $\lambda < 0$, in this case $X(x) = C_0 e^{\sqrt{-\lambda}x} + C_1 e^{\sqrt{-\lambda}x}$ is the general solution and using $X(0) = 0$ we have $C_0 = -C_1$ and using $X(1) = 0$ we get $C_0 = C_1 = 0$. Hence we get trivial solution.

Now we know that only non-trivial solution comes from when $\lambda = k^2 > 0$ for some $k > 0$ (again when $\lambda = 0$ and $\lambda < 0$ will give only the trivial solution, $X(x) = 0$). In this case the solution is

$$X(x) = A\cos(kx) + B\sin(kx).$$

We now find A and B using the boundary values we have in (b), at $x = 0$

$$X(0) = A\cos(0) + B\sin(0) = 0 \quad \text{implies} \quad A = 0.$$

At $x = 1$, (now $A = 0$, we only have $X(x) = B\sin(kx)$)

$$X(0) = B\sin(k) = 0.$$

This holds if $\sin(k) = 0$ which gives $k = \pi n$. Hence $k = \pi n$.

$$k = \pi n \quad \text{hence} \quad \lambda = k^2 = \pi^2 n^2.$$

Therefore, the eigenvalues are

$$\lambda_n = \pi^2 n^2 \quad n = 1, 2, \ldots.$$

The corresponding eigenfunctions are

$$X_n(x) = \sin(kx) = \sin(\pi n x) \quad n = 1, 2, \ldots.$$

(d) (6 points) For each eigenvalue λ_n you found in (c), solve the initial value problem corresponding to T_n.

Solution: Since $\lambda_n = \pi^2 n^2$ and the ordinary differential equation corresponding to T is

$$T'' + 9\lambda T = 0.$$

Substitute $\lambda_n = \pi^2 n^2$ we get (we have a different solution for each n);

$$T_n'' + 9\lambda T_n = T_n'' + 9\pi^2 n^2 T_n = 0.$$

This is a second order linear differential equation and which has characteristic equation

$$r^2 + 3^2\pi^2 n^2 = 0.$$

From this we get that the characteristic equation has a imaginary complex conjugate roots

$$r = \pm 3\pi n\mathbf{i}.$$

Thus the solutions are

$$T_n(t) = A_n \cos(3\pi n t) + B_n \sin(3\pi n t) \quad n = 1, 2, \ldots, .$$

(e) (2 points) Now write general solution for each n, $u_n(x,t) = X_n(x)T_n(t)$ and find the general solution $u(x,t) = \sum u_n(x,t)$.
Solution: Combining (c) and (d) we have

$$u_n(x,t) = X_n(x)T_n(t) = \sin(\pi n x)[A_n \cos(3\pi n t) + B_n \sin(3\pi n t)] \quad n = 1, 2, \ldots, .$$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \sin(\pi n x)[A_n \cos(3\pi n t) + B_n \sin(3\pi n t)].$$

(f) (8 points) Using the given initial values, find the particular solution.
Solution: Using the given initial condition $u(x,0) = 3435 sin(\pi x) + 2018\sin(5\pi x)$ and evaluating the solution we found in the previous part at $t = 0$ we get

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\pi n x) = 3435 sin(\pi x) + 2018\sin(5\pi x).$$

From this we see that $A_1 = 3435$ and $A_5 = 2018$ and all other $A_n = 0$ (we do not know what B_n's are yet). To figure out B_n's we use the second initial condition (keep A_n as they are for the moment).

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin(\pi n x)[-A_n 3\pi n \sin(3\pi n t) + B_n 3\pi n \cos(3\pi n t)].$$

and at $t = 0$ we get

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin(\pi n x) B_n 3\pi n = 0 \quad \text{which implies } B_n = 0 \ , n = 1, 2, \ldots .$$

Hence (remember $A_1 = 3435$ and $A_5 = 2018$ and all other $A_n = 0$)

$$u(x,t) = 3435\sin(\pi x)\cos(3\pi t) + 2018\sin(5\pi x)\cos(15\pi t)$$

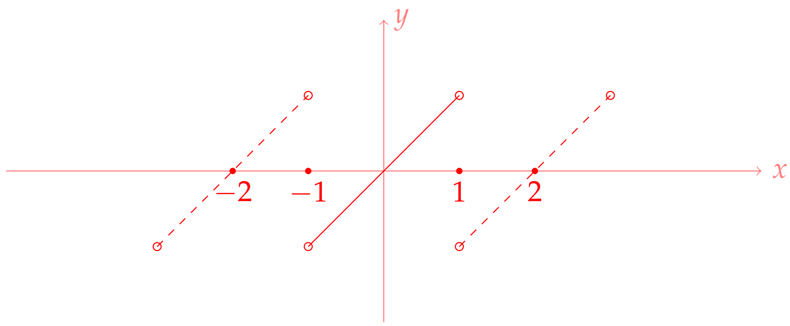
is the solution we are looking for.

4. Let $f(x)$ be given as

$$f(x) = x, \quad 0 \le x < 1.$$

(a) (6 points) Extend $f(x)$ into an odd periodic function with period of 2.

Solution: As we want to extend f into odd periodic function with period of 2 we define



Odd extension $F_{\text{odd}}(x)$ of f is, $F_{\text{odd}}(x+2) = F_{\text{odd}}(x)$ and

$$F_{\text{odd}}(x) = \begin{cases} f(x) & 0 \le x < 1, \\ -f(-x) & -1 < x \le 0. \end{cases} = \begin{cases} x & 0 \le x < 1, \\ -(-x) & -1 < x \le 0 \end{cases}$$

(b) (6 points) Find Fourier series $F(x)$ of the periodic function you found in (a).

Solution: As we have the extension F_{odd}, which is periodic with period of $2L = 2$, (hence $L = 1$) we will find its Fourier series.

Since we want odd extension, then all even terms or equivalently cosine terms will be zero ($a_0 = 0$ and $a_n = 0$). Hence it will be a sine series and we need to find b_n where

$$\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} F_{\text{odd}}(x)\sin(\frac{n\pi x}{L})dx \\
&= \int_{-1}^{1} F_{\text{odd}}(x)\sin(\frac{n\pi x}{1})dx \\
&= 2\int_{0}^{1} F_{\text{odd}}(x)\sin(\frac{n\pi x}{1})dx \\
&= 2\int_{0}^{1} x\sin(n\pi x)dx \\
&= -\frac{2}{n\pi}[x\cos(n\pi x)]_{x=0}^{x=1} + \frac{2}{n\pi}\int_{0}^{1}\cos(n\pi x)dx \\
&= -\frac{2}{n\pi}\cos(n\pi) + \frac{2}{n^2\pi^2}\sin(n\pi x)|_{x=0}^{x=1} \\
&= -\frac{2}{n\pi}\cos(n\pi).
\end{aligned}$$

Notice that $\cos(n\pi)$ is 1 when n is even and is -1 when n is odd. Hence we get $\cos(n\pi) =$

$(-1)^n$. We have $b_n = 2(-1)^{n+1}/(n\pi)$. Hence the Fourier series of F_{odd} is

$$\begin{aligned} F(x) &= \sum_{n=1}^{\infty} b_n \sin(n\pi x) \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n\pi} \sin(n\pi x). \end{aligned}$$

(c) (3 points (bonus)) Using part (a)-(b) verify that

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

Solution: Since $F_{\text{odd}}(1/2) = 1/2 = F(1/2)$ as F_{odd} is continuous and differentiable around that point it agrees with its Fourier series at that point. Using this we get

$$\begin{aligned} F_{\text{odd}}(1/2) = 1/2 = F(1/2) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n\pi} \sin(n\pi x) \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n\pi} \sin(n\pi/2) \end{aligned}$$

Using this and the fact that $\sin(n\pi/2)$ is 0 when n is even and 1 or -1 when n is odd. Rewriting this we get

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

5. (12 points) Solve the first order equation

$$2u_x + 3u_y = 0$$

with the auxiliary condition

$$u(x,0) = \frac{1}{1+e^x}.$$

Solution: Notice that

$$\langle (2,3), \nabla u(x,y) \rangle = 2u_x + 3u_y = 0$$

Hence $u(x,y)$ is constant in the direction of $\langle 2,3 \rangle$. The lines parallel to $\langle 2,3 \rangle$ have equations $-3x+2y=c$. Here $-3x+2y=c$ is called the characteristic lines. As $u(x,y)$ is constant on these lines therefore $u(x,y)$ depends only $-3x+2y$. Hence

$$u(x,y) = f(-3x+2y).$$

Using the the auxiliary condition $u(x,0) = \frac{1}{1+e^x}$ we get

$$u(x,0) = f(-3x) = \frac{1}{1+e^x}.$$

Since $f(-3x) = \frac{1}{1+e^x}$. We can find $f(x) = \frac{1}{1+e^{-x/3}}$. Since

$$u(x,y) = f(-3x+2y) = \frac{1}{1+e^{-(-3x+2y)/3}}.$$

is the solution.

6. True of False, you need to **justify your answer** why it is true of false. You will not get credit if you only write true or false.

   (a) (6 points) The Fourier series of

   $$\begin{aligned} f(x) = & \, 1977 + 1980\sin(x) + 1983\cos(x) \\ & + 1999\sin(2x) + 2002\cos(2x) + 2005\sin(3x) \\ & + 2015\cos(3x) + 2017\sin(4x) + 2019\cos(4x) \end{aligned}$$

   is itself.

   Solution: True, since every term is already a sine or a cosine term and is periodic. Hence the Fourier series of $f(x)$ is itself.

   (b) (6 points) The Fourier series of

   $$f(x) = x^4 + \cos(3410x), \quad -\pi \le x \le \pi, \quad f(x+2\pi) = f(x)$$

   is a Fourier **sine** series.

   Solution: False. Since $f(x)$ is even function, it should be a cosine series. In fact, all sine terms will be zero.

   (c) (8 points) The function $u(x,t) = \sin(2x)\cos(4t)$ is a solution to the following equation with boundary conditions

   $$4u_{xx} = u_{tt} \quad \text{with} \quad u(0,t) = 0 \text{ and } u(\pi,t) = 0.$$

   Solution: True. Since $u_{xx} = -4\sin(2x)\cos(4t)$ and $u_{tt} = -16\sin(2x)\cos(4t)$. Hence $4u_{xx} = u_{tt}$. Moreover,

   $$u(0,t) = \sin(0)\cos(4t) = 0 \quad \text{and} \quad u(\pi,t) = \sin(2\pi)\cos(4t) = 0.$$

7. (20 points (bonus)) Consider the following partial differential equation

$$u_x + 2xu_y = 0.$$

Using separation of variables method (follow the steps in question 2 or question 3), find general solution $u(x, y)$ of the form $u(x, y) = X(x)Y(y)$.

Solution: We use th hint which tells us that there is a separable solution $u(x, y) = X(x)Y(y)$. As

$$u_x = X'Y \quad \text{and} \quad u_y = XY'$$

and writing above PDE in terms of X and Y we get

$$X'Y + 2xXY' = 0 \quad \text{or} \quad \frac{X'}{2xX} = -\frac{Y'}{Y}.$$

One side is a function of x and the other side is a function of y, we know this is possible only if they are constant. Say

$$\frac{X'}{2xX} = -\frac{Y'}{Y} = -\lambda \quad \text{for some constant } \lambda.$$

From this we get

$$\frac{X'}{2xX} = -\lambda \quad \text{hence} \quad X' + 2x\lambda X = 0$$
$$-\frac{Y'}{Y} = -\lambda \quad \text{hence} \quad Y' - \lambda Y = 0.$$

Focus on the first equation; $X' + 2x\lambda X = 0$ equivalently, $X'/X = -2x\lambda$. If we integrate both sides we get

$$\ln X = -x^2\lambda + c \quad \text{or} \quad X(x) = Ae^{-\lambda x^2}.$$

For the second equation we have $Y'/Y = \lambda$ which has solution

$$Y(y) = Be^{\lambda y}.$$

Hence general solution is

$$u(x, y) = X(x)Y(y) = Ae^{-\lambda x^2}Be^{\lambda y}.$$