UCONN - Math 3410 - Fall 2017 - Solution to Graded Problems of HW3

Question 1 (Problem 1, 5 Points) *Find the orthogonal trajectories of the family of circles centered on x-axis and passing through the origin.*

Solution: This family of curves has equation

$$(x-k)^2 + y^2 = k^2.$$

Now we need to find a 1-parameter family of curves orthogonal to above family of curves. To find this find the slope y' of above curves at arbitrary points (x, y);

$$2(x-k) + 2yy' = 0.$$

Now we need to get rid of k by solving above and once you solve for k you get

$$k = \frac{x^2+y^2}{2x}.$$

Substitute this into the DE to get

$$2(x - \frac{x^2+y^2}{2x}) + 2yy' = 0$$

Now solve for y';

$$y' = -\frac{x^2-y^2}{2xy}.$$

As we are looking for 1-parameter family of curves which is orthogonal to above family, the slope of the curves at arbitrary points we are looking for should"-reciprocal of y' "

$$y' = \frac{2xy}{x^2-y^2} \quad \text{or} \quad (x^2-y^2)dy - 2xydx = 0$$

As in our notations; $Mdx + Ndy = 0$ where $M = -2xy$ and $N = x^2 - y^2$. One needs to solve this DE. This is not a exact differential equation as $M_y = -2x$ and $N_x = 2x$ are not equal to each other. Now From the theorem we covered in class; as $p(y) = (N_x - M_y)/M = 4x/(-2xy) = -2/y$ is independent of x we get $e^{\int p(y)dy}$ is an integrating factor, i.e, y^{-2} is an integrating factor with $y \neq 0$. Now multiply above DE to get an exact DE;

$$y^{-2}(x^2-y^2)dy - 2\frac{x}{y}dx = 0.$$

Now this is an exact DE then, following the steps, integrate $-2x/y$ with respect to x to find the solution

$$f(x,y) = -\frac{x^2}{y} + g(y) = 0$$

is a solution to the DE. Using $f_y = \frac{x^2}{y^2} - 1 = \frac{x^2}{y^2} + g'(y)$ we get $g(y) = -y + c$ and thereupon

$$-\frac{x^2}{y} - y + c = 0 \quad \text{or by multiplying everything with } -y$$

one gets

$$x^2 + y^2 = cy$$

which is the 1-parameter family of curves orthogonal to $(x-k)^2 + y^2 = k^2$.

Question 2 (Problem 5, 5 points) *Find the general solution of the Ricatti equation*

$$y' = 1 + \frac{y}{x} - \frac{y^2}{x^2}$$

with given particular solution $y_1(x) = x$.

Solution: We know that the change of variable is $y = y_1(x) + \frac{1}{u} = x + \frac{1}{u}$ will convert the equation into a nice one. Hence,

$$y' = 1 - \frac{1}{u^2}u'.$$

Substitute this into the DE to get

$$\begin{aligned} 1 - \frac{1}{u^2}u' &= 1 + \frac{x + \frac{1}{u}}{x} - \frac{(x + \frac{1}{u})^2}{x^2} \\ &= 1 + 1 + \frac{1}{ux} - \frac{x^2 + \frac{2x}{u} + \frac{1}{u^2}}{x^2} \\ &= 1 + 1 + \frac{1}{ux} - 1 - \frac{2}{ux} - \frac{1}{u^2x^2}. \end{aligned}$$

Hence if we do some algebra we first get

$$-\frac{1}{u^2}u' = -\frac{1}{ux} - \frac{1}{u^2x^2}$$

and multiplying everything by $-u^2$ to get

$$u' - \frac{u}{x} = \frac{1}{x^2}.$$

This can be written as

$$(\frac{u}{x})' = \frac{1}{x^3}.$$

Integrating this one gets

$$\frac{u}{x} = \frac{-1}{2x^2} + c \quad \text{equivalently} \quad u = \frac{-1}{2x} + cx.$$

Hence the solution we are looking for is

$$y(x) = x + \frac{1}{u} = x + \frac{1}{\frac{-1}{2x} + cx}.$$

* Thanks for Junhao for the correction.