UCONN - Math 3410 - Fall 2017 - Solution to Graded Problems of HW4

Question 1 (Question 25-3, 5 Points) *Find the inverse Laplace transform of* $F(s) = \frac{s^2+1}{s^2(s+2)}$.

Solution: Since

$$\frac{s^2+1}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2}.$$

After some algebra one can find $A = -1/4$, $B = 1/2$, and $C = 5/4$. Hence

$$\mathcal{L}^{-1}(\frac{s^2+1}{s^2(s+2)}) = \mathcal{L}^{-1}(\frac{-1/4}{s} + \frac{1/2}{s^2} + \frac{5/4}{s+2}).$$

Since Laplace transform is linear we also have

$$\mathcal{L}^{-1}(\frac{s^2+1}{s^2(s+2)}) = -1/4\mathcal{L}^{-1}(\frac{1}{s}) + 1/2\mathcal{L}^{-1}(\frac{1}{s^2}) + (5/4)\mathcal{L}^{-1}(\frac{1}{s+2}).$$

Therefore,

$$\mathcal{L}^{-1}(\frac{s^2+1}{s^2(s+2)}) = -1/4 + (1/2)x + (5/4)e^{-2x}.$$

Question 2 (Question 28, 5 Points) *Use the Laplace transform to solve the following initial value problem*

$$y'' + y = \cos(2x) \quad \textit{with } y(0) = 2,\ y'(0) = 1.$$

Solution: Using Laplace transform we get

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{\cos(2x)\}.$$

Using the properties we have

$$s^2\mathcal{L}\{y(x)\} - sy(0) - y'(0) + \mathcal{L}\{y(x)\} = \frac{s}{s^2+4}.$$

Now combine $\mathcal{L}\{y(x)\}$ in one side and move everything to the other side to get

$$(s^2+1)\mathcal{L}\{y(x)\} = \frac{s}{s^2+4} + 2s + 1.$$

Then

$$\mathcal{L}\{y(x)\} = \frac{1}{s^2+1}\frac{s}{s^2+4} + \frac{2s+1}{s^2+1}.$$

Equivalently,

$$\mathcal{L}\{y(x)\} = \frac{1}{s^2+1}\frac{s}{s^2+4} + 2\frac{s}{s^2+1} + \frac{1}{s^2+1}.$$

Now we have to write

$$\frac{1}{s^2+1}\frac{s}{s^2+4} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}.$$

After some algebra one gets $A = 1/3$, $C = -1/3$, $B = D = 0$. Hence

$$\mathcal{L}\{y(x)\} = (1/3)\frac{s}{s^2+1} + (-1/3)\frac{s}{s^2+4} + 2\frac{s}{s^2+1} + \frac{1}{s^2+1}.$$

Now we can find inverse Laplace transform of both sides to get

$$\begin{aligned} y(x) &= \mathcal{L}^{-1}\{(1/3)\frac{s}{s^2+1} + (-1/3)\frac{s}{s^2+4} + 2\frac{s}{s^2+1} + \frac{1}{s^2+1}\} \\ &= (1/3)\mathcal{L}^{-1}\{\frac{s}{s^2+1}\} + (-1/3)\mathcal{L}^{-1}\{\frac{s}{s^2+4}\} + 2\mathcal{L}^{-1}\{\frac{s}{s^2+1}\} + \mathcal{L}^{-1}\{\frac{1}{s^2+1}\} \\ &= \frac{7}{3}\cos(x) + \sin(x) - \frac{1}{3}\cos(2x)\,. \end{aligned}$$