UCONN - Math 3410 - Fall 2017 - Solution to Graded Problems of HW5

Question 1 (Question 1, 5 Points) *Find a power series solution $y(x)$ around the point $x_0 = 0$ to the differential equation $y'' + y = 0$. Verify that the power series solution you found has the form $y(x) = a_0 \cos(x) + a_1 \sin(x)$.*

Solution: Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Since $p(x) = 1 = q(x)$ are both analytic at $x_0 = 0$ then the solution will be analytic at $x_0 = 0$ with radius of convergence ∞. Find y'' to get

$$\begin{aligned}
0 = y'' + y &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + a_n]x^n.
\end{aligned}$$

Therefore, the corresponding recurrence relation is $(n+2)(n+1)a_{n+2} + a_n = 0$ for every $n \geq 0$. Now, consider the even terms first;

$$\begin{aligned}
a_2 &= \frac{-a_0}{2\,1}, \\
a_4 &= \frac{-a_2}{4\,3} = \frac{a_0}{4\,3\,2} \\
a_6 &= \frac{-a_4}{6\,5} = \frac{-a_0}{6\,5\,4\,3\,2} \\
&\cdots \\
a_{2n} &= (-1)^n \frac{a_0}{(2n)!} \text{ for } n = 1, 2, \ldots
\end{aligned}$$

Similarly, odd terms are

$$\begin{aligned}
a_3 &= \frac{-a_1}{3\,2}, \\
a_5 &= \frac{-a_1}{5\,4} = \frac{a_1}{5\,4\,3\,2} \\
a_7 &= \frac{-a_5}{7\,6} = \frac{-a_1}{7\,6\,5\,4\,3\,2} \\
&\cdots \\
a_{2n+1} &= (-1)^n \frac{a_1}{(2n+1)!} \text{ for } n = 1, 2, \ldots
\end{aligned}$$

Now the solution $y(x)$ can be written as even terms plus the odd terms

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}.$$

Hence

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} a_{2k}x^{2k} + \sum_{k=0}^{\infty} a_{2k+1}x^{2k+1} \\ &= \sum_{k=0}^{\infty}(-1)^k \frac{a_0}{(2k)!}x^{2k} + \sum_{k=0}^{\infty}(-1)^k \frac{a_1}{(2k+1)!}x^{2k+1}. \end{aligned}$$

One can check that these power series have radius of convergence infinity, (i.e., converges for every x). Remember that $\sin(x)$ and $\cos(x)$ have power series representation

$$\sin(x) = \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)!}x^{2k+1} \quad \text{and} \quad \cos(x) = \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k)!}x^{2k}.$$

Now using these we get

$$y(x) = a_0 \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k)!}x^{2k} + a_1 \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)!}x^{2k+1} = a_0\cos(x) + a_1\sin(x).$$

Question 2 (Question 3, 5 Points) *Consider the Rayleigh's equation*

$$my'' + ky = ay' - b(y')^3$$

which models the oscillation of a clarinet reed. Using the second method find the first four terms of the power series solution $y(x)$ around $x_0 = 0$ with $m = k = a = 1$ and $b = 1/3$ with the initial conditions $y(0) = 0$ and $y'(0) = 1$. Write the solution $y(x)$.

Solution: It is given that $m = k = a = 1$ and $b = 1/3$, plugin these numbers into the differential equation to get

$$y'' + y = y' - \frac{1}{3}(y')^3 \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 1.$$

Since $x_0 = 0$ is ordinary point, $y(x)$ has Taylor series expansion around $x_0 = 0$;

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!}x^n, \quad \text{i.e.,} \quad a_n = \frac{y^{(n)}(0)}{n!}.$$

Therefore, using the second method, we find $y(0), y'(0), y''(0), y'''(0)$. Luckily, $y(0) = 0$ and $y'(0) = 1$ are given. So we need to find $y''(0)$. Use the differential equation, plugin $x = 0$ to get

$$y''(0) + y(0) = y'(0) - \frac{1}{3}(y'(0))^3 \quad \text{equivalently} \quad y''(0) = 1 - \frac{1}{3} = \frac{2}{3}.$$

Now we need to find $y'''(0)$. To do this end, differentiate the differential equation with respect to x and evaluate at $x = 0$ to get

$$y'''(0) + y'(0) = y''(0) - (y'(0))^2 y''(0) \quad \text{equaivalently} \quad y'''(0) = -1.$$

Hence

$a_0 = 0$ $\qquad$ $a_1 = 1$ $\qquad$ $a_2 = \frac{2}{3}\frac{1}{2!}$ $\qquad$ $a_3 = \frac{-1}{3!}$.

$$y(x) = 0 + x - \frac{1}{3}x^2 + \frac{1}{6}x^3 + \dots$$