UCONN - Math 3410 - Fall 2017 - HW6
Solutions to graded problems

1. [**Problem 1**] For the following differential equation $4xy'' + 2y' + y = 0$.

   - Find and classify all points as ordinary, regular singular, or irregular singular points.
   - For each of the regular point(s), find the corresponding indicial equation and find roots r_1 and r_2 of the indicial equation (Yes, there are two roots and the difference is not integer).
   - Find the corresponding recurrence relations for each of the roots r_1, r_2.
   - Find the corresponding power series solutions y_1 and y_2.

   Solution: Rewrite the differential equation as

   $$y'' + \frac{2}{4x}y' + \frac{1}{4x}y = 0.$$

   Then $p(x) = 2/4x$ and $q(x) = 1/4x$ both have singularities at $x = 0$. Therefore, all points except $x = 0$ is ordinary points. For $x = 0$ we need to check the following limits;

   $$\lim_{x\to 0} xp(x) = \lim_{x\to 0} \frac{2x}{4x} = 1/2 \quad \text{and} \quad \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{1}{4x} = 0.$$

   Since both limits exist and are finite, therefore $x = 0$ is regular singular points.

   Since $x = 0$ is the only regular singular point, we find the corresponding indicial equation. Let

   $$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

   be a solution for some r and a_n. Find y' and y'' in terms of power series.

   $$y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}.$$

   Plug in to the differential equation to get

   $$4x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + 2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

   That is,

   $$\sum_{n=0}^{\infty} 4(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

   In order to write the sums under one sum, we need to change the power of x in the last summation from $n+r$ to $n+r-1$. Therefore,

   $$\sum_{n=0}^{\infty} 4(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0.$$

Now if we split $n = 0$ terms in the first two sums and leave the rest under summation we get

$$4r(r-1)a_0x^{r-1}+\sum_{n=1}^{\infty}4(n+r)(n+r-1)a_nx^{n+r-1}+2ra_0x^{r-1}+\sum_{n=1}^{\infty}2(n+r)a_nx^{n+r-1}+\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}=0.$$

Now we have (combining x^{r-1} and remaining terms under big sum)

$$(4r(r-1)+2r)a_0x^{r-1}+\sum_{n=1}^{\infty}[4(n+r)(n+r-1)a_n+2(n+r)a_n+a_{n-1}]x^{n+r-1}=0$$

Since this is true every x, we get

$$(4r(r-1)+2r)a_0=0 \quad \text{and} \quad 4(n+r)(n+r-1)a_n+2(n+r)a_n+a_{n-1}=0 \quad \text{for every} \quad n\geq 1. \tag{1}$$

The first identity gives us (assuming $a_0 \neq 0$) the indicial equation $4r(r-1)+2r=0$. Therefore, the roots are $r_1 = 1/2$ and $r_2 = 0$. We have two distinct roots and $r_1 - r_2$ is not integer.

Then the method of Frobenious implies that we have two linearly independent solution y_1 corresponding to r_1 and y_2 corresponding to r_2

$$y_1(x)=\sum_{n=0}^{\infty}a_nx^{n+\frac{1}{2}} \quad \text{and} \quad y_2(x)=\sum_{n=0}^{\infty}b_nx^{n+0}=\sum_{n=0}^{\infty}b_nx^n.$$

Plug $r_1 = 1/2$ and solve for a_n in (1) to get

$$a_n=\frac{-a_{n-1}}{(2n+1)2n}$$

which gives

$$\begin{aligned}
a_1 &= \frac{-a_0}{3\,2}=\frac{-a_0}{3!}\\
a_2 &= \frac{-a_1}{5\,4}=\frac{a_0}{5\,4\,3!}=\frac{a_0}{5!}\\
a_3 &= \frac{-a_2}{7\,6}=\frac{-a_0}{7\,6\,5!}=\frac{-a_0}{7!}\\
&\dots\\
a_n &= (-1)^n\frac{a_0}{(2n+1)!} \quad n=1,2,\dots,
\end{aligned}$$

Therefore we get the first solution corresponding to $r_1 = 1/2$.

$$y_1(x)=\sum_{n=0}^{\infty}a_nx^{n+\frac{1}{2}}=a_0\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}x^{n+\frac{1}{2}}$$

To find the solution corresponding to $r_2 = 0$, plug in $r_2 = 0$ in (1) (this time i am writing the recurrence relation in terms of b_n) to get

$$b_n=\frac{-b_{n-1}}{2n(2n-1)}.$$

Now (assuming $b_0 = a_0 \neq 0$)

$$\begin{aligned}
b_1 &= \frac{-b_0}{2\,1} = \frac{-a_0}{2!} \\
b_2 &= \frac{-b_1}{4\,3} = \frac{b_0}{4\,3\,2!} = \frac{a_0}{4!} \\
b_3 &= \frac{-b_2}{6\,5} = \frac{-b_0}{6\,5\,4!} = \frac{-a_0}{6!} \\
&\cdots \\
b_n &= (-1)^n \frac{b_0}{(2n)!} \quad n = 1, 2, \ldots,
\end{aligned}$$

We get the second linearly independent solution corresponding to $r_2 = 0$

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n = b_0 \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^n.$$

2. [**Problem 2**] For the following differential equation $xy'' + y' - y = 0$.
   - Find and classify all points as ordinary, regular singular, or irregular singular points.
   - For each of the regular point(s), find the corresponding indicial equation and find the double root r_1 of the indicial equation (Yes there is one double root).
   - Find the corresponding recurrence relation for the root r_1.
   - Find the corresponding power series solution y_1.
   - Use the method of Frobenious and write down the general form of the second solution y_2.
   - Find at least first two terms b_0 and b_1 of the second solution y_2.

Solution: Rewrite the differential equation as

$$y'' + \frac{y'}{x} - \frac{y}{x} = 0$$

Therefore, $p(x) = 1/x$ and $q(x) = -1/x$ are both singular at $x = 0$. We conclude that all points except $x = 0$ is ordinary points. For $x = 0$ we need to check the following limits

$$\lim_{x\to 0} xp(x) = \lim_{x\to 0} \frac{x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{1}{x} = 0.$$

Since both limits exist and are finite, therefore $x = 0$ is regular singular points.

Since $x = 0$ is the only regular singular point, we find the corresponding indicial equation. Let

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

be a solution for some r and a_n. Find y' and y'' in terms of power series.

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}.$$

Plug in to the differential equation to get

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}-\sum_{n=0}^{\infty}a_nx^{n+r}=0.$$

After some algebra one gets

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-1}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}-\sum_{n=0}^{\infty}a_nx^{n+r}=0.$$

In order to write the three summation under one sum we need to change the power of x from $n+r$ to $n+r-1$ so that the powers of x in each summation match (one can change the power of x from $n+r-1$ to $n+r$ in the first two summation, idea is the same). Therefore,

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-1}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}-\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}=0.$$

We can rewrite the power series as (just split the $n=0$ terms and leave the remaining)

$$r(r-1)a_0x^{r-1}+\sum_{n=1}^{\infty}(n+r)(n+r-1)a_nx^{n+r-1}+ra_0x^{r-1}+\sum_{n=1}^{\infty}(n+r)a_nx^{n+r-1}-\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}=0.$$

This gives us

$$(r(r-1)a_0+ra_0)x^{r-1}+\sum_{n=1}^{\infty}[(n+r)(n+r-1)a_n+(n+r)a_n-a_{n-1}]x^{n+r-1}=0$$

Since this is true for every x we get (assuming again $a_0\neq 0$)

$$r(r-1)+r=0 \quad \text{and} \quad (n+r)(n+r-1)a_n+(n+r)a_n-a_{n-1}=0 \quad \text{for} \quad n\geq 1. \tag{2}$$

Therefore the indicial equation is $r^2=0$. Hence we have a double root $r_1=0$.

To find the corresponding recurrence relation corresponding to $r_1=0$, simply plug in $r=0$ in (2) to get

$$a_n=\frac{a_{n-1}}{n^2} \quad \text{for} \quad n\geq 1.$$

As we assume that $a_0\neq 0$ we get

$$\begin{aligned}
a_1 &= a_0,\\
a_2 &= \frac{a_1}{2^2}=\frac{a_0}{2^2}=\frac{a_0}{(2!)^2}\\
a_3 &= \frac{a_2}{3^2}=\frac{a_0}{3^2 2^2}=\frac{a_0}{(3!)^2}\\
&\dots\\
a_n &= \frac{a_0}{(n!)^2}.
\end{aligned}$$

Therefore, we obtain the first solution corresponding to $r_1 = 0$

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} = a_0 \sum_{n=0}^{\infty} \frac{1}{(n!)^2} x^n.$$

Since the indicial equation has double root, the method of Frobenious tells us that the second linearly independent solution is of the form

$$y_2(x) = \sum_{n=1}^{\infty} b_n x^{n+r} + \log(x) y_1(x) = \sum_{n=0}^{\infty} b_n x^n + \log(x) y_1(x)$$

Without plug in $y_1(x)$, we find $y_2'(x)$ and $y_2''(x)$. To this end, let us write $Y(x) = \sum_{n=1}^{\infty} b_n x^n$ and

$$y_2(x) = \sum_{n=1}^{\infty} b_n x^{n+r} + \log(x) y_1(x) = Y(x) + \log(x) y_1(x).$$

Now

$$y_2'(x) = Y'(x) + \frac{y_1(x)}{x} + \log(x) y_1'(x) \quad \text{and} \quad y_2''(x) = Y''(x) + \frac{y_1'(x)}{x} - \frac{y_1(x)}{x^2} + \frac{y_1'(x)}{x} + \log(x) y_1''(x).$$

Plug in this into the differential equation to get

$$xy_2'' + y_2' - y = xY'' + y_1' - \frac{y_1}{x} + y_1' + x\log(x) y_1'' + Y' + \frac{y_1}{x} + \log(x) y_1' - Y - \log(x) y_1 = 0$$

Observe that we have the following terms $x\log(x) y_1'' + \log(x) y_1' - \log(x) y_1 = \log(x)[xy_1'' + y_1' - y_1] = 0$ as y_1 is a solution to the above differential equation. Therefore we have

$$xy_2'' + y_2' - y_2 = xY'' + y_1' - \frac{y_1}{x} + y_1' + Y' + \frac{y_1}{x} - Y = xY'' + Y' - Y + 2y_1' = 0 \quad (3)$$

Note that $Y(x) = \sum_{n=1}^{\infty} b_n x^n$, therefore one has

$$Y'(x) = \sum_{n=1}^{\infty} n b_n x^{n-1} \quad \text{and} \quad Y''(x) = \sum_{n=2}^{\infty} n(n-1) b_n x^{n-2}$$

Plug in these into (3) to get (and move $2y_1'$ to right hand side

$$\begin{aligned} 2y_1' = xY'' + Y' - Y &= \sum_{n=2}^{\infty} n(n-1) b_n x^{n-1} + \sum_{n=1}^{\infty} n b_n x^{n-1} - \sum_{n=1}^{\infty} b_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1) b_n x^{n-1} + \sum_{n=1}^{\infty} n b_n x^{n-1} - \sum_{n=2}^{\infty} b_{n-1} x^{n-1} \\ &= \sum_{n=2}^{\infty} [n(n-1) b_n + n b_n - b_{n-1}] x^{n-1} + b_1. \end{aligned}$$

Note that since $y_1(x) = a_0 \sum_{n=0}^{\infty} \frac{1}{(n!)^2} x^n = a_0 x - a_0 \frac{x^2}{4} + \ldots$ we get

$$-2y_1'(x) = -2a_0 + a_0 x + \ldots = \sum_{n=2}^{\infty} [n(n-1) b_n + n b_n - b_{n-1}] x^{n-1} + b_1$$

from which we get $b_1 = -2a_0$. To find b_2, we set $n = 2$ in the summation to get $(4b_2 - b_1)x = a_0 x$ and therefore $b_2 = -3a_0/4$.

3. [**Problem 3**] For the following differential equation $xy'' + y = 0$.

   - Find and classify all points as ordinary, regular singular, or irregular singular points.
   - For each of the regular point(s), find the corresponding indicial equation and find the roots r_1 and r_2 of the indicial equation (Yes there are two roots with $r_1 - r_2$ is integer).
   - Find the corresponding recurrence relation for the root r_1.
   - Find the corresponding power series solution y_1.
   - Use the method of Frobenious and write down the general form of the second solution y_2.
   - Find at least first two terms b_0 and b_1 of the second solution y_2.

   Solution: Rewrite the differential equation as

$$y'' + \frac{1}{x}y = 0.$$

Then $p(x) = 0$ and $q(x) = 1/x$ both have singularities at $x = 0$. Therefore, all points except $x = 0$ is ordinary points. For $x = 0$ we need to check the following limits;

$$\lim_{x\to 0} xp(x) = \lim_{x\to 0} x0 = 0 \quad \text{and} \quad \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{1}{x} = 0.$$

Since both limits exist and are finite, therefore $x = 0$ is regular singular points.

Since $x = 0$ is the only regular singular point, we find the corresponding indicial equation. Let

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

be a solution for some r and a_n. Find y' and y'' in terms of power series.

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}.$$

Then plug into the differential equation to get

$$x\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

Now we have

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

and changing the power of x in the second summation to get

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

and this is equal to

$$r(r-1)a_0 x^{r-1} + \sum_{n=1}^{\infty} [(n+r)(n+r-1)a_n + a_{n-1}] x^{n+r-1} = 0 \tag{4}$$

From this we get the indicial equation $r(r-1)=0$, which gives us $r_1=1$ and $r_2=0$ (here it does not matter if you choose $r_1=0$ and $r_2=1$ and proceed accordingly).

Plug in $r=1$ into the recurrence relation in (4) to get

$$(n+1)(n+1-1)a_n + a_{n-1} = 0 \quad \text{and} \quad \text{for} \quad n \geq 1.$$

From this we get

$$a_n = \frac{-a_{n-1}}{(n+1)n}$$

Now

$$\begin{aligned}
a_1 &= \frac{-a_0}{2\,1} \\
a_2 &= \frac{-a_1}{3\,2} = \frac{a_0}{3\,2\,2\,1} = \frac{a_0}{3\,(2!)^2} \\
a_3 &= \frac{-a_2}{4\,3} = \frac{-a_0}{4\,(3!)^2} \\
&\dots \\
a_n &= \frac{(-1)^n a_0}{(n+1)(n!)^2}.
\end{aligned}$$

Since $r_1 - r_2$ is an integer, the method of Frobenious tells us that the first solution corresponding to $r_1 = 1$ (it does not matter if you chose $r_1 = 0$ and proceed) is

$$y_1(x) = x\sum_{n=0}^{\infty} a_n x^n = a_0 x \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n!)^2} x^n = a_0 x - a_0\frac{x^2}{2} + a_0\frac{x^3}{12} + \dots$$

The method of Frobenious tells us that the second solution is of the form

$$y_2(x) = x^0 \sum_{n=0}^{\infty} b_n x^x + c y_1(x)\log(x)$$

and following as in the second problem one gets $b_0 = -ca_0$ and $2b_2 + b_1 = (3/2)a_0 c$.