UCONN - Math 3410 - Fall 2017 - Problem Set 7
Solutions to graded problems

Problem 2: Let $f(x)$ be given as

$$f(x) = \begin{cases} x & 0 \le x \le \pi/2, \\ \pi - x & \pi/2 \le x \le \pi. \end{cases}$$

a. (4pt) Extend $f(x)$ into an odd periodic function with period of 2π and find its Fourier series $F(x)$.

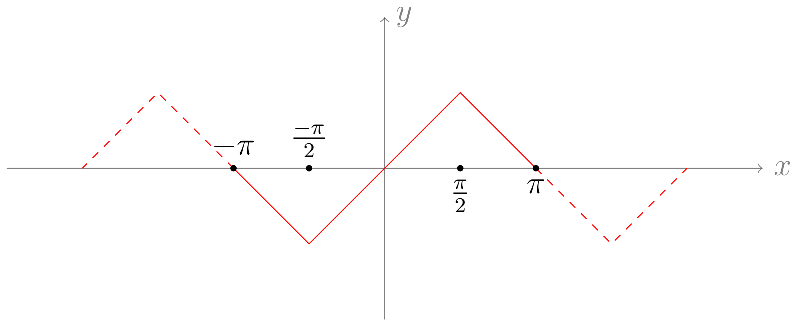
b. (4pt) Extend $f(x)$ into an even periodic function with period of 2π and find its Fourier series $F(x)$.

c. (2pt) Use either part (a) or part (b) to verify that

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

Solution:

- Odd extension of $f(x)$



Odd extension $F_{\text{odd}}(x)$ of f is, $F_{\text{odd}}(x + 2\pi) = F_{\text{odd}}(x)$ and

$$F_{\text{odd}}(x) = \begin{cases} f(x) & 0 \le x \le \pi, \\ -f(-x) & -\pi \le x \le 0. \end{cases} = \begin{cases} x & 0 \le x \le \pi/2, \\ \pi - x & \pi/2 \le x \le \pi \\ -(-x) & -\pi/2 \le x \le 0, \\ -(\pi - (-x)) & -\pi \le x \le \pi/2. \end{cases}$$

As we have the extension F_{odd}, which is periodic with period of $2L = 2\pi$, we will find its Fourier series.

Since this is an odd extension, cosine terms will be zero; $a_n = 0$ and $a_0 = 0$. Hence it will be a sine series; we only need to find b_n.

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} F_{\text{odd}} \sin(nx)dx = \frac{2}{\pi}\int_{0}^{\pi} F_{\text{odd}} \sin(nx)dx \\
&= \frac{2}{\pi}\int_{0}^{\pi/2} x\sin(nx)dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x)\sin(nx)dx \\
&= \frac{2}{\pi}[-\frac{x\cos(nx)}{n}|_0^{\pi/2} + \frac{1}{n}\int_{0}^{\pi/2}\cos(nx)dx] + \frac{2}{\pi}[-\frac{(\pi-x)\cos(nx)}{n}|_{\pi/2}^{\pi} - \frac{1}{n}\int_{\pi/2}^{\pi}\cos(nx)dx] \\
&= \frac{2}{\pi}[0 + \frac{1}{n^2}\sin(nx)|_0^{\pi/2} + 0 - \frac{1}{n^2}\sin(nx)|_{\pi/2}^{\pi}] \\
&= \frac{4}{\pi n^2}\sin(n\pi/2).
\end{aligned}
$$

Hence $b_n = 0$ when n is even and therefore we only have odd terms; (set $n = 2k-1$)

$$b_{2k-1} = \frac{4}{\pi n^2}\sin((2k-1)\pi/2) = \frac{4}{\pi}\frac{(-1)^{k+1}}{(2k-1)^2} \text{ for } k = 1, 2, \dots$$

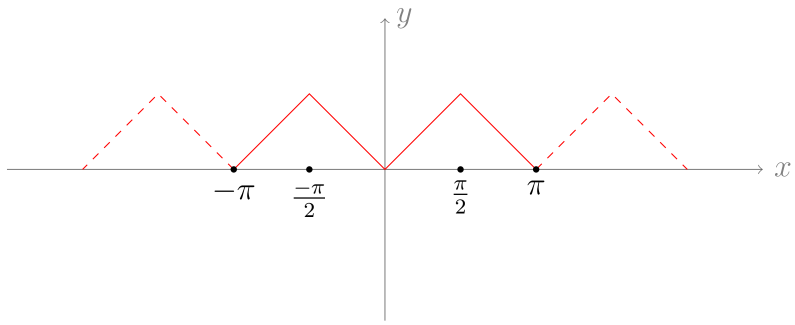
Hence, the Fourier series of $F_{\text{odd}}(x)$ (notice that the odd terms in b_k is zero, we only have the odd terms) is

$$0 + 0 + \sum_{k=1}^{\infty} b_k \sin(kx) = \sum_{k=1}^{\infty}\frac{4}{\pi}\frac{(-1)^{k+1}}{(2k-1)^2}\sin((2k-1)x)$$

Note that $F_{\text{odd}}(x)$ is continuous everywhere. Hence by the Fourier series convergence theorem we know that $F_{\text{odd}}(x)$ and its Fourier series agree for every x;

$$F_{\text{odd}}(x) = \sum_{k=1}^{\infty}\frac{4}{\pi}\frac{(-1)^{k+1}}{(2k-1)^2}\sin((2k-1)x).$$

- Even extension of $f(x)$



Odd extension $F_{\text{even}}(x)$ of f is, $F_{\text{even}}(x + 2\pi) = F_{\text{even}}(x)$ and

$$F_{\text{even}}(x) = \begin{cases} f(x) & 0 \le x \le \pi, \\ f(-x) & -\pi \le x \le 0. \end{cases} = \begin{cases} x & 0 \le x \le \pi/2, \\ \pi - x & \pi/2 \le x \le \pi \\ (-x) & -\pi/2 \le x \le 0, \\ (\pi - (-x)) & -\pi \le x \le \pi/2. \end{cases}$$

As we have the extension F_{even}, which is periodic with period of $2L = 2\pi$, we will find its Fourier series.

Since this is an even extension, sine terms will be zero; $b_n = 0$. Hence it will be a cosine series; we only need to find a_n and a_0. Let us find a_0 first;

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F_{\text{even}}(x)dx = \frac{2}{\pi}\int_{0}^{\pi} F_{\text{even}}(x)dx = \frac{2}{\pi}\int_{0}^{\pi/2} xdx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)dx = \frac{2}{\pi}[\frac{\pi^2}{8} + \frac{\pi^2}{8}] = \frac{\pi}{2}.$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} F_{\text{even}}(x)\cos(nx)dx = \frac{2}{\pi}\int_{0}^{\pi} F_{\text{even}}(x)\cos(nx)dx \\
&= \frac{2}{\pi}\int_{0}^{\pi/2} x\cos(nx)dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\cos(nx)dx \\
&= \frac{2}{\pi}[\frac{x\sin(nx)}{n}|_0^{\pi/2} - \frac{1}{n}\int_{0}^{\pi/2}\sin(nx)dx] + \frac{2}{\pi}[\frac{(\pi - x)\sin(nx)}{n}|_{\pi/2}^{\pi} + \frac{1}{n}\int_{\pi/2}^{\pi}\sin(nx)dx] \\
&= \frac{2}{\pi}[\frac{\pi}{2n}\sin(n\pi/2) + \frac{1}{n^2}\cos(n\pi/2) - \frac{1}{n^2}] + \frac{2}{\pi}[\frac{-\pi}{2n}\sin(n\pi/2) - \frac{1}{n^2}\cos(n\pi) + \frac{1}{n^2}\cos(n\pi/2)] \\
&= \frac{4}{\pi n^2}\cos(n\pi/2) - \frac{2}{\pi n^2} - \frac{2}{\pi n^2}\cos(n\pi) = \frac{4}{\pi n^2}\cos(n\pi/2) - \frac{2}{\pi n^2} - \frac{2}{\pi n^2}(-1)^n
\end{aligned}$$

Hence $a_n \neq 0$ only when $n = 2k + 4$ for $k = 0, 1, \ldots$. From this observation one gets Fourier series of F as

$$\frac{\pi}{4} - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos((4k+2)x).$$

With the same reasoning as above, F is continuous everywhere, hence the Fourier series of $F_{\text{even}}(x)$ and the function $F_{\text{even}}(x)$ agree for every x;

$$F_{\text{even}}(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos((4k+2)x).$$

- Now we use either of the Fourier series we found to see that, for example use the first one and evaluate at $x = \pi/2$ to get

$$\frac{\pi}{2} = F_{\text{odd}}(\pi/2) = \sum_{k=1}^{\infty}\frac{4}{\pi}\frac{(-1)^{k+1}}{(2k-1)^2}\sin((2k-1)\pi/2).$$

  A little algebra gives

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$