UCONN - Math 3410 - Fall 2017 - Problem Set 8
Solutions to graded problems

Problem 2: Consider the following Heat conduction problem

$$\begin{cases} 9u_{xx} = u_t, \quad 0 < x < 3, \quad t > 0 \\ u_x(0,t) = 0 \quad \text{and} \quad u_x(3,t) = 0 \\ u(x,0) = 2\cos(\frac{\pi x}{3}) - 4\cos(\frac{5\pi x}{3}). \end{cases}$$

1. By considering separation of variables $u(x,t) = X(x)T(t)$, rewrite the partial differential equation in terms of two ordinary differential equations in X and T (take arbitrary constant as $-\lambda$).
2. Rewrite the boundary values in terms of X and T. **Be careful on the boundary conditions for** u_x.
3. Now choose the boundary values which will not give a non-trivial solution and write the ordinary differential equation corresponding to X.
4. Solve the two-point boundary value problem corresponding to X. Find all eigenvalues λ_n and eigenfunctions X_n.
5. For each eigenvalue λ_n you found in (d), rewrite and solve the ordinary differential equation corresponding to T_n.
6. Now write general solution for each n, $u_n(x,t) = X_n(x)T_n(t)$ and find the general solution $u(x,t) = \sum u_n(x,t)$.
7. Using the given initial value and the general solution you found in (f), find the particular solution.

Solution

- By considering separation of variables $u(x,t) = X(x)T(t)$, rewrite the partial differential equation in terms of two ordinary differential equations in X and T (take arbitrary constant as $-\lambda$).

Solution: Rewrite the PDE as $9u_{xx} - u_t = 0$. Let $u(x,t) = X(x)T(t)$. Then

$$u_{xx} = X''T \quad \text{and} \quad u_t = XT'.$$

Substitute this in to the differential equation to get

$$9u_{xx} - u_t = 9X''T - XT' = 0 \quad \text{equivalently} \quad \frac{X''}{X} = \frac{T'}{9T} = -\lambda.$$

Hence

$$\frac{X''}{X} = -\lambda \quad \rightarrow \quad X'' + \lambda X = 0,$$
$$\frac{T'}{9T} = -\lambda \quad \rightarrow \quad T' + 9\lambda T = 0.$$

- Rewrite the boundary values in terms of X and T. **Be careful on the boundary conditions for u_x.**

Solution: We have at $x = 0$ as $u_x(x,t) = X'(x)T(t)$ then

$$u_x(0,t) = X'(0)T(t) = 0; \quad \text{one has either} \quad X'(0) = 0 \quad \text{or} \quad T(t) = 0.$$

At $x = \pi$

$$u_x(3,t) = X'(\pi)T(t) = 0; \quad \text{one has either} \quad X'(3) = 0 \quad \text{or} \quad T(t) = 0.$$

- Now choose the boundary values which will not give a non-trivial solution and write the ordinary differential equation corresponding to X.

Solution: We know that the choice of $T(t) = 0$ gives only the trivial solution as $u(x,t) = X(x)T(t) = 0$.

Therefore, we choose our boundary conditions as $X'(0) = 0$ and $X'(3) = 0$ in order to obtain the non-trivial solution. Now if we rewrite the ordinary differential equation corresponding to X we get

$$X'' + \lambda X = 0, \quad X'(0) = 0 \quad \text{and} \quad X'(3) = 0.$$

- Solve the two-point boundary value problem corresponding to X. Find all eigenvalues λ_n and eigenfunctions X_n.

For $\lambda = 0$, the ordinary differential equation $X'' + \lambda X = 0$ becomes $X'' = 0$. Then the solution is

$$X(x) = ax + b \quad \text{for some constants} \quad a, b.$$

Using the boundary conditions $X'(0) = 0 \quad$ and $\quad X'(3) = 0$ we get $X'(x) = a$ and then $X'(0) = a = 0$. Hence we have $a = 0$. The second boundary conditions is also verified as $X'(3) = a = 0$. Hence $X(x) = b$ is solution corresponding to the eigenvalue $\lambda = 0$.

For $\lambda < 0$. Say $\lambda = -n^2$ for some constant $n > 0$. Then

$$X'' + \lambda X = X'' - n^2 X = 0.$$

This differential equation has characteristic equations $r^2 - n^2 = 0$, hence roots are $r = \pm n$. This gives the solution

$$X(x) = Ae^{nx} + Be^{-nx}.$$

Take derivative to get

$$X'(x) = Ane^{nx} - Bne^{-nx}.$$

Using boundary conditions we get

$$X'(0) = Ane^0 - Bne^0 = 0 \quad \text{and} \quad X'(3) = Ane^{3n} - Bne^{3n} = 0$$

which leads us to $X(x) = 0$. Hence we get the trivial solution.

For $\lambda > 0$. Let $\lambda = k^2$ for some constant $k > 0$, we get

$$X'' + \lambda X = X'' + k^2 X = 0.$$

We know that the solution is

$$X(x) = A\cos(kx) + B\sin(kx).$$

Using this general solution and the first boundary condition and $X'(x) = -Ak\sin(kx) + Bk\cos(kx)$ that

$$X'(0) = -Ak\sin(0) + Bk\cos(0) = Bk = 0 \quad \text{therefore we have} \quad B = 0.$$

Using the second boundary condition, we get (as $B = 0$, $X'(x) = -Ak\sin(kx)$)

$$X'(\pi) = -Ak\sin(3k) = 0.$$

This holds when $3k = n\pi$ for $n = 1, 2, \ldots$. Hence we get $k = n\pi/3$, or equivalently, we get the eigenvalues

$$\lambda_n = k^2 = \frac{n^2\pi^2}{3^2}, \quad n = 1, 2, \ldots.$$

The corresponding eigenfunction (corresponding to λ_n) is

$$X_n(x) = \cos(\frac{n\pi x}{3}).$$

- For each eigenvalue λ_n you found in (d), rewrite and solve the ordinary differential equation corresponding to T_n.

Solution: We have for $\lambda = 0$, $X_0(x) = b$ as a nontrivial solution. Hence we need to find corresponding solution T_0 for $\lambda = 0$. When $\lambda = 0$, the corresponding differential equation is

$$T' + 9\lambda T = T' = 0.$$

$T' = 0$ has solution $T(t) = b$ for some constant b. Hence for $\lambda = 0$ we have corresponding solution $T_0(t) = b$.

For $\lambda = \frac{n^2\pi^2}{3^2} > 0$ the ordinary differential equation corresponding to T is now

$$T' + 9\lambda T = 0.$$

Plug in $\lambda = \frac{n^2\pi^2}{3^2}$ we get (for each n we have a different solution T_n)

$$T_n' + 9\frac{n^2\pi^2}{3^2}T_n = T_n' + n^2\pi^2 T_n = 0.$$

We know that this is a first order linear ordinary differential equation and its solution is

$$T_n(t) = C_n e^{-n^2\pi^2 t}.$$

for some C_n.

- Now write general solution for each n, $u_n(x,t) = X_n(x)T_n(t)$ and find the general solution $u(x,t) = \sum u_n(x,t)$.

Solution: We know that the solution for each $n > 0$ is

$$u_n(x,t) = X_n(x)T_n(t) = \cos(\frac{n\pi x}{3})C_n e^{-n^2\pi^2 t}.$$

and when $n = 0$ we have $u_0(x,t) = ab = C_0$ for some constant C_0. The general solution is

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} u_n(x,t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(\frac{n\pi x}{3})e^{-n^2\pi^2 t}.$$

- Using the given initial value and the general solution you found in (f), find the particular solution.

Now the initial value is given as $u(x,0) = 2\cos(\frac{\pi x}{3}) - 2\cos(\frac{5\pi x}{3})$. Hence, plug in $t = 0$ in the solution we have in (f) gives us

$$u(x,0) = C_0 + \sum_{n=1}^{\infty} C_n \cos(\frac{n\pi x}{3})e^0 = C_0 + \sum_{n=1}^{\infty} C_n \cos(\frac{n\pi x}{3}) = 2\cos(\frac{\pi x}{3}) - 2\cos(\frac{5\pi x}{3}).$$

From this we get that, $C_0 = 0$, and all $C_n = 0$ except $C_1 = 2$ and $C_5 = -2$. Hence in the solution, the only terms we have, for $n = 1, n = 5$ with $C_1 = 2$, $C_5 = -2$;

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} C_n \cos(\frac{n\pi x}{3})e^{-n^2\pi^2 t} \\ &= 2\cos(\frac{\pi x}{3})e^{-1^2\pi^2 t} - 2\cos(\frac{5\pi x}{3})e^{-5^2\pi^2 t} \\ &= 2\cos(\frac{\pi x}{3})e^{-\pi^2 t} - 2\cos(\frac{5\pi x}{3})e^{-25\pi^2 t}. \end{aligned}$$