UCONN - Math 3410 - Fall 2017 - Problem Set 9
Solutions to graded problems

Question 2: Consider the following wave equation which describes the displacement $u(x,t)$ of a piece of flexible string with the initial boundary value problem

$$\begin{cases} 25u_{xx} = u_{tt}, \quad 0 < x < 5, \quad t > 0, \\ u(0,t) = 0 \quad \text{and} \quad u(5,t) = 0, \\ u(x,0) = 0 \quad \text{and} \quad u_t(x,0) = 3\sin(\frac{3\pi x}{5}) - 10\sin(\frac{4\pi x}{5}). \end{cases}$$

1. By considering separation of variables $u(x,t) = X(x)T(t)$, rewrite the partial differential equation in terms of two ordinary differential equations in X and T (take arbitrary constant as $-\lambda$).
2. Rewrite the boundary values in terms of X and T.
3. Now choose the boundary values which will not give a non-trivial solution and write the ordinary differential equation corresponding to X.
4. Solve the two-point boundary value problem corresponding to X. Find all eigenvalues λ_n and eigenfunctions X_n.
5. For each eigenvalue λ_n you found in (d), rewrite and solve the ordinary differential equation corresponding to T_n.
6. Now write general solution for each n, $u_n(x,t) = X_n(x)T_n(t)$ and find the general solution $u(x,t) = \sum u_n(x,t)$.
7. Using the given initial value and the general solution you found in (f), find the particular solution.

Solution

- By considering separation of variables $u(x,t) = X(x)T(t)$, rewrite the partial differential equation in terms of two ordinary differential equations in X and T (take arbitrary constant as $-\lambda$).

Solution: Rewrite the differential equation as $25u_{xx} - u_{tt} = 0$. Let $u(x,t) = X(x)T(t)$. Then we get

$$u_{xx} = X''T \quad \text{and} \quad u_{tt} = XT''.$$

Substitute this into the partial differential equation $25u_{xx} - u_{tt} = 0$ to get

$$25u_{xx} - u_{tt} = 25X''T - XT'' = 0.$$

Dividing by $25XT$ we get

$$\frac{X''}{X} = \frac{T''}{25T}.$$

Notice that the left-hand side is a function of x only and the right-hand side is function of t only and as they are same, this is possible only if they are the same constant;

$$\frac{X''}{X} = \frac{T''}{25T} = -\lambda.$$

From this we get

$$\frac{X''}{X} = -\lambda \quad \to \quad X'' + \lambda X = 0$$
$$\frac{T''}{4T} = -\lambda \quad \to \quad T'' + 25\lambda T = 0.$$

- Rewrite the boundary values in terms of X and T.

Since $u(x,t) = X(x)T(t)$ we have when $x = 0$

$$u(0,t) = X(0)T(t) = 0 \quad \text{we should have either} \quad X(0) = 0 \quad \text{or} \quad T(t) = 0.$$

At $x = 5$, we have

$$u(5,t) = X(5)T(t) = 0 \quad \text{we should have either} \quad X(5) = 0 \quad \text{or} \quad T(t) = 0.$$

- Now choose the boundary values which will not give a non-trivial solution and write the ordinary differential equation corresponding to X.

We know that $T(t) = 0$ will give only the trivial solution. Therefore, we should choose

$$X(0) = 0 \quad \text{and} \quad X(5) = 0.$$

Then X satisfies the following two-point boundary condition

$$X'' + \lambda X = 0, \quad X(0) = 0 \quad \text{and} \quad X(5) = 0.$$

- Solve the two-point boundary value problem corresponding to X. Find all eigenvalues λ_n and eigenfunctions X_n.

Now we know that only non-trivial solution comes from when $\lambda = k^2$ for some $k > 0$ (again when $\lambda = 0$ and $\lambda < 0$ will give only the trivial solution, $X(x) = 0$). In this case the solution is

$$X(x) = A\cos(kx) + B\sin(kx).$$

We now find A and B using the boundary values we have in (c), at $x = 0$

$$X(0) = A\cos(0) + B\sin(0) = 0 \quad \text{implies} \quad A = 0.$$

At $x = 5$, (now $A = 0$, we only have $X(x) = B\sin(kx)$)

$$X(0) = B\sin(5k) = 0.$$

This holds if $\sin(5k) = 0$ which gives $5k = \pi n$. Hence $k = \pi n/5$.

$$k = \frac{\pi n}{5} \quad \text{hence} \quad \lambda = k^2 = \frac{\pi^2 n^2}{5^2}.$$

Therefore, the eigenvalues are

$$\lambda_n = \frac{\pi^2 n^2}{5^2}.$$

The corresponding eigenfunctions are

$$X_n(x) = \sin(kx) = \sin(\frac{\pi n x}{5})$$

- For each eigenvalue λ_n you found in (d), rewrite and solve the ordinary differential equation corresponding to T_n.

Since $\lambda_n = \frac{\pi^2 n^2}{5^2}$ and the ordinary differential equation corresponding to T is

$$T'' + 25\lambda T = 0.$$

Substitute $\lambda =$ we get (we have a different solution for each n);

$$T_n'' + 25\lambda T_n = T_n'' + 25\frac{\pi^2 n^2}{5^2} T_n = T_n'' + \pi^2 n^2 T_n = 0$$

This is a second order linear differential equation and which has characteristic equation

$$r^2 + \pi^2 n^2 = 0.$$

From this we get that the characteristic equation has a imaginary complex conjugate roots

$$r = \pm \pi n \mathbf{i}.$$

Thus the solutions are

$$T_n(t) = A_n \cos(\pi n t) + B_n \sin(\pi n t) \quad n = 1, 2, \dots, .$$

- Now write general solution for each n, $u_n(x,t) = X_n(x)T_n(t)$ and find the general solution $u(x,t) = \sum u_n(x,t)$.

Combining (c) and (e) we have

$$u_n(x,t) = X(x)T(t) = \sin(\frac{\pi n x}{5})[A_n \cos(\pi n t) + B_n \sin(\pi n t)].$$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \sin(\frac{\pi n x}{5})[A_n \cos(\pi n t) + B_n \sin(\pi n t)].$$

- Using the given initial value and the general solution you found in (f), find the particular solution.

Now we have two initial values $u(x,0) = 0$ and $u_t(x,0) = 3\sin(\frac{3\pi x}{5}) - 10\sin(\frac{4\pi x}{5})$
At $t = 0$, i.e, $u(x,0) = 0$

$$u(x,0) = \sum_{n=1}^{\infty} \sin(\frac{\pi n x}{5})[A_n \cos(0) + B_n \sin(0)] = 0$$

gives us $A_n = 0$ for every $n = 1, 2, \dots$. Hence we have

$$u(x,t) = \sum_{n=1}^{\infty} \sin(\frac{\pi n x}{5}) B_n \sin(\pi n t).$$

we now need to find $u_t(x,t)$;

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin(\frac{\pi n x}{5}) B_n \pi n \cos(\pi n t).$$

Then given initial value $u_t(x,0) = 3\sin(\frac{3\pi x}{5}) - 10\sin(\frac{4\pi x}{5})$ gives us

$$u_t(x,0) = \sum_{n=1}^{\infty} \sin(\frac{\pi n x}{5}) B_n \pi n \cos(0) = 3\sin(\frac{3\pi x}{5}) - 10\sin(\frac{4\pi x}{5}).$$

Therefore, let $b_n = B_n \pi n$ then above identity becomes

$$\sum_{n=1}^{\infty} b_n \sin(\frac{\pi n x}{5}) = 3\sin(\frac{3\pi x}{5}) - 10\sin(\frac{4\pi x}{5}).$$

This tells us all $b_n = 0$ except $b_3 = 3$ and $b_4 = -10$ as

$$\sum_{n=1}^{\infty} b_n \sin(\frac{\pi n x}{5}) = b_1 \sin(\frac{\pi x}{5}) + b_2 \sin(\frac{\pi 2x}{5}) + b_3 \sin(\frac{\pi 3x}{5}) + b_4 \sin(\frac{\pi 4x}{5}) + \dots$$

and we know that this summation is $3\sin(\frac{3\pi x}{5}) - 10\sin(\frac{4\pi x}{5})$. Hence all b_n should be zero except b_3 which will be 3 and b_n which will be -10. As $b_n = B_n \pi n$, then we get $b_3 = B_3 \pi 3 = 3$, which gives us $B_3 = \frac{1}{\pi}$. Similarly, $b_4 = B_4 \pi 4 = -10$, which gives us $B_4 = \frac{-10}{4\pi}$. Hence

$$u(x,t) = B_3 \sin(\frac{\pi 3x}{5}) \sin(3\pi t) + B_4 \sin(\frac{\pi 4x}{5}) \sin(4\pi t) = \frac{1}{\pi} \sin(\frac{\pi 3x}{5}) \sin(3\pi t) + \frac{-10}{4\pi} \sin(\frac{\pi 4x}{5}) \sin(4\pi t).$$